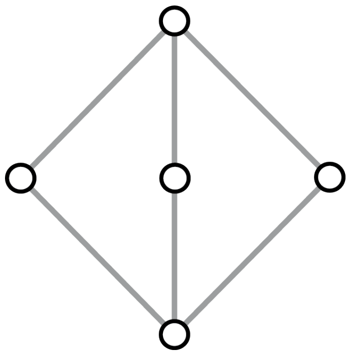
# An Inquiry into Dialectic Logic

Juan Grompone



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# Dedication

This book is dedicated to the memory of cherished teachers who have been unknowingly responsible for its coming into being.

When I began my university studies, I was strongly drawn by mathematics. This led me to enroll in an elementary symposium organized by the Mathematics Institute at the Faculty of Engineering. In one of the first meetings, José Luis Massera asked me which branch of mathematics I was most interested in. I timidly replied "mathematical logic", which was my interest back then, when I was utterly unaware of the breadth of mathematics. Massera laughed, as he used to, and his words became forever imprinted on my memory. I will quote them as I remember them.

> When I am old and senile, I will grow a little beard and devote myself to mathematical logic.

Such was his categorical and contemptuous regard for the topic. That is why, now, over half a century later–having returned to my primary calling–I would like to dedicate this book to the memory of José Luis Massera, my esteemed teacher. Perhaps he would not have approved of my attempt at making a formal proposition on dialectics, but I have no doubt that he would have read it with some curiosity. I am sure he would have let out a laugh or two, with his characteristic Homeric laughter.

My second teacher was Fernando Forteza. I began to make a formalization of dialectics towards the end of the 1970s, amidst a military dictatorship in full swing. I had been removed from my position at the University and was banned from the Faculty of Engineering, which meant I did not have access to the Library of the Mathematics Institute or any bibliography on lattices, which is the natural setting for logic. I once ran into Forteza and asked him, rather shyly, if I could borrow Birkhoff's [4] book, which could be found at the library. A couple of days later he brought it to me and I quickly made a copy before returning it. That book was fundamental to the present study.

My third teacher was Mario H. Otero, who took a chance on these *Enquiries* –as well as other intellectual adventures–, publishing them in the journal "Galileo" of the Faculty of Humanities and Sciences, despite the exotic nature of the subject matter within the field of epistemology. This book is also a celebration of his memory and intellectual generosity.

# Contents

# Foreword

This book is divided into three sections. In the first one, the use of dialectics is substantiated by means of examples taken from the natural languages, spontaneous forms of dialectic thought and relevant historical cases. This section ends with the chapter entitled "Intuitive introduction", which contains no mathematical resources nor makes any formalization. The second section begins in the following chapter, "Formalization", and carries on until the second-to-last chapter. This section makes use of algebraic resources and formalizes the theory. The final chapter applies dialectics within the realm of formal, experimental and social sciences. In summary, readers who wish to skip the mathematical formulation may go from the "Intuitive introduction" straight to the final chapter. If you have any doubts on the meaning of the concepts used, please refer to the analytical index.

The dialectic logic studied in this book is a multi-valued extension of binary logic. It is a mathematical structure defined within a lattice containing a group of automorphisms and anti-automorphisms. The double nature of this formal structure makes it extremely rich in terms of its properties and potential applications.

These inquiries are the result of a synthesis which attempts to settle traditional binary logic, a somewhat informal version of Hegel's dialectic, and the structures of spontaneous human thought which are not entirely accessible or comprehensible through traditional binary logic.

I believe this fourth revision of the work has gained much in terms of clarity and with regards to practical applications. An increase in the speed of computers is responsible for one major difference with the previous versions: it allows to study functions in a more systematic manner. However, there is still much work to be done, since more complex lattices call for greater computation times.

Rather than being merit of my own, Rafael Grompone is to be thanked for most of the adjustments: he was patient enough to carefully review this book and object, discuss and suggest modifications, some of which have yet to be added to the present version. Another major contribution is that of Lucía Grompone, who reviewed and corrected the diagrams, proposing a coherent style so the book could have the best possible presentation. I am very grateful to both of them. However, this cannot be considered a final version. There are many aspects which remain unexplored, as well as some obvious issues and notorious gaps. The following have been omitted from this version:

- slightly modifying the mathematical notation of negations so as to preserve lattice symmetry with regards to the central values– see the note in page 120;
- systematically analyzing logical functions in lattices with $r > 3$;
- expanding the study of dialectic quantifiers, which are barely outlined in this version;
- completing the dialectic study of the paradoxes cited in the corresponding chapter;
- analyzing Battro-Piaget's formalization on operational logic, see [2, III, 1].

These will be considered for future versions of this document, as well as any other observations and amendments as they may be sent to the email addresses which are available for this purpose.

Montevideo, July of 2017.

In *www.grompone.org* are published the C programs that allows to study the dialectical functions for the lattices used in this book.

# Logic as an image of the universe

## Introduction

Human knowledge is a giant accumulation of statements. These statements make up collections and groups, and have structures which help connect them. These connections between statements are *logical* in nature. The full collection of statements–or any partial set of these–may be considered as an *algebraic* structure capable of being *analyzed and characterized*.

The universe of *statements* is a miscellaneous collection which may belong to any of the possible categories of human thought. For the science of logic, a logical structure may not only occur within the realm of mathematics. Anywhere we can acknowledge a certain "coherence"–that is to say, a certain formal structure–we bear witness to one of the manifestations of logical thought. Therefore, for example, we must accept the following:

- the statements made by pre-Socratic Greek thinkers, especially Heraclitus (-535?, -475?), his contemporaries, or classical Chinese philosophers;
- statements related to quantum mechanics–and other areas of science–due to their peculiar "irrationality";
- spontaneous Aymara statements–and the use of Spanish in some areas of America–which may lead to exotic logical structures;
- jokes, paradoxes, poetry, as long as we recognize a formal value in them and not a mere play on words;
- in the same sense, esoteric and astrological statements, as well as those from the Kabbalah.

The science of logic is concerned with the "structure" of human thought. For this reason, we must begin by searching for a structure capable of handling statements used in a natural and spontaneous manner.

## The universe of statements

If we were to make a formal examination, we must say there are different categories of statements. To begin with, we must distinguish between simple and complex statements. Simple statements look like this:

> Amos Judd loves cold mutton.[1]
>
> Socrates is mortal.[2]

We are not very much interested–at least for the time being–in defining simple statements with much precision. In many cases, this character depends on the manner in which the statement is analyzed. For the purposes of the science of logic, this topic is of little importance.

On the other hand, complex statements are formed by means of simple statements and *logical connectives*.

Beyond their specificities, all the natural languages possess elements which allow them to formulate logical statements. In Latin-derived languages, there are ways to present the negation function as well as basic functions of two variables of binary logic. These functions are expressed by means of *conjunctions*. In some cases, punctuation signs replace elliptical conjunctions, which is a very widely used literary resource, as we will see.

Linguists classify conjunctions according to criteria which do not always agree with the logic. They call *copulative* conjunctions those corresponding to the **AND** operation (or its negation). An example would be:

> God is dead. Marx is dead. **AND** I don't feel so well myself.[3]

[1] A fantastic statement by Lewis Carrol which can be found in [11].

[2] A classical statement that should never be missing from any work of logic. I do not know who the author is and so confess from the very first pages of this book.

[3] Eugène Ionesco. Wikiquotes.

In this operation, the logical conjunction **AND** is at once associative and commutative and does not present major difficulties. As in the example provided, most languages allow the use of a *comma* to express the operation in a repeated manner. This use of the comma also extends to other cases.

*Disjunctive* conjunctions correspond to the logical operation **OR**. Some basic examples are:

Freedom **OR** death.[4]

By reason **OR** force.[5]

The logical disjunction introduces some difficulties. It is also associative and commutative. The comma is also usually employed to repeatedly apply the function. In general, in Latin-derived languages, there is no doubt as to the symmetry of the disjunction operation.[6]

In the examples of disjunction we can always doubt whether the speaker is referring to an operation of inclusion or one of exclusion. Sometimes, when dismissing the potential ambiguity is desired, one can do so explicitly, by saying:

Freedom **OR** death, *or both*.

*Distributive* conjunctions are referred to as the various conjunctions corresponding to the excluding logical function. Its forms are quite diverse in the different languages. In general, it is necessary to clarify the meaning (let us put on hold for a moment the use of "but"):

Freedom **OR** death, *but not both*.

Conjunctions expressing the different forms of logical *implication* are referred to as conditionals, *concessive*, *inferential*, and many other terms. Perhaps this multiplicity of names and ways of referring to them point to something that is still unknown to us. Taking a typical mathematical statement:

---

[4] Phrase coined by Juan Antonio Lavalleja upon liberating the current territory of Uruguay from Portuguese power.

[5] The motto which appears in the coat of arms of the Republic of Chile.

[6] The English language displays an oddity with regards to the disjunction. The form *either* … *or* suggests that there is no symmetry in this operation.

**if** $x$ **implies** $y$ **then** $z$.

Negations and negative statements present many difficulties. It can be said, in all fairness, that the science of logic is devoted to resolving this issue. This is why we will not insist on this matter at this moment.

Without attempting to make an exhaustive list, let us remember the statements of existence:

**Some** oysters are silent.[7]

And universal statements:

**All** men are mortals.[8]

For over 25 centuries, there has been a preoccupation with classifying, formalizing and interpreting these statements and logical connectives. There was a huge leap forward in the $19^{th}$ century, when George Boole (1815, 1864) discovered the first formal properties of these structures. In the first decades of the $20^{th}$ century it was considered that the entire formalization had been completed. Bertrand Russell (1872, 1970) proved that, for instance, negation and disjunction were sufficient to construct all the remaining logical connectives. He was also eloquent in his thesis in that only two quantifiers related to each other–existential and universal–were capable of describing everything that was needed with regards to statements in the natural languages.

There is good reason to believe, however, that there are logical structures which escape this fairly simple scenario. Hidden as commas or other punctuation signs, concealed in logical connectives not easily identifiable, there may be logical functions which elude the simple logical universe described by Russell.

## Ambivalent statements

Natural languages use *ambivalent* words extemporaneously. Some examples may illustrate this curious property. We will begin by an am-

[7] Other fantastic statement by Lewis Carrol which can be found in [11].

[8] Another classical statement no work of logic should ever do without.

bivalence which is common to languages of Indo-European origin: sexual intercourse.

In Latin there is the verb "*futuere*", and all the derived expressions: in French, "*foutre*", in Catalan, "*fotre*", in Italian, "*fottere*", in Spanish, "*joder*", and in Portuguese, "*foder*" The Germanic languages have: in English, "to fuck", in German, "*ficken*", in Dutch, "*fokken*", in Norwegian, "*fukka*", in Swedish, "*focka*". In almost all cases, the verb is ambivalent: sexual intercourse, damage or struggle.

In Spanish, the verb "*joder*" is clearly ambivalent. On the one side, it refers to intercourse, something essential to the preservation of the species and, therefore–as Darwin will explain–a something pleasant. However, the word also means the exact opposite: to deceive, to hurt.[9] Its use reaches the greatest contradiction by means of the reflexive expression "*jódete*".

We must not think that this peculiarity only applies to Spanish. Other Latin languages also share this ambivalence. In English, the verb "to fuck" has practically the same usage.[10] In Germanic languages, the ambivalence is almost always verified. It is possible that in some cases this aspect of the word has become obsolete.

There are other expressions which are also ambivalent. If we take the Spanish phrase, "*de puta madre*"[11], the expression is used both as a strong *insult* and an equally strong *praise*.

Leaving sexual aspects aside, in American Spanish, words such as "*brutal*", "*bestial*"[12] or "*soberbio*",[13] which according to the dictionary

[9] The *Spanish Language Dictionary* (DLE) [19] gives it the following meanings: to engage in sexual intercourse, to be irritated or fooled, to sexually possess a woman, to annoy or bother someone, to destroy, ruin or let something go to waste.

[10] The *New Oxford American Dictionary* awards the following meanings: have sexual intercourse with (someone), ruin or damage (something). The same happens with the various associated verbal phrases, including the reflexive version "fuck yourself".

[11] The word "*puta*" is ambivalent in Spanish. According to the DLE [19]: a derogatory qualifier, to ponder, to emphasize the absence or shortage of something. In this case, the meaning is three-fold: negative, positive and neutral.

[12] According to the DLE [19]: brutal or irrational, of disproportionate grandeur or extraordinary.

[13] According to the DLE [19]: arrogant or being driven by arrogance; grandiose or magnificent.

are negative qualifiers, can also be positive, and it is only the context which allows to make this distinction. The same happens with the qualifier "*arrecho*" o "*verraco*".[14]

The French language has a rhetorical form of expression known as "*litote*"[15] which sets forth a thesis while meaning to say the exact opposite of the formulated statement.[16] An example might be the expression *il n'est pas complètement stupide* (he is not completely stupid). The literal meaning is that the person is sometimes not stupid. In its rhetorical sense, the phrase means that he "is very smart".[17]

The oxymoron[18] is directly linked to this rhetorical figure. Some classical examples are: "*Festina lente*" (hurry up slowly), from the emperor Augustus; "feather of lead, bright smoke, cold fire, sick health" from Shakespeare (Romeo and Juliet); "una graciosa torpeza" (a gracious clumsiness), from Jorge Luis Borges (*The aleph*). This sums up twenty centuries of using this contradicting rhetoric.

Finally, paradoxes are complex statements that contradict themselves. This point will be analyzed further in pages to come.

What is the logic behind using ambivalent words–or expressions–with contradicting meanings?[19] Dialectics have an explanation for this: the law of penetration of opposites (see page 52). Some actions or

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[14] According to the DLE [19], "*arrecho*" applies to a person, depending on the region of America: sexually aroused, angry or furious, brave or spirited, lucky, spectacular or sensational, strongly vehement, very difficult. The DLE does not include the two meanings for "*verraco*" which in the Caribbean refers to something despicable, very large or bad, but being a qualifier which also expresses admiration.

[15] All languages possess statements of this kind, in French they are of everyday use.

[16] The word comes from the Greek λιτοτης (*litotes*, simplicity), but is also a figure of classical rhetoric in which it is hinted that the meaning is not as simple as it seems. It is a beautiful example of Greek dialectics.

[17] Texts which are not originally in English have been translated from the author's own Spanish translations.

[18] This word is a Latin neologism from the $5^{th}$ century, comprised by οξυς (*oxys*, acute, smart) and μωρος (*moros*, fool, stupid). "Oxymoron" is an oxymoron.

[19] In languages with a Latin influence there are two words that derive from "*contrarius*" and "*oppositus*". They are usually close in meaning but also have some differences. The identity of these two words is a basic notion in dialectics. In German, Engels [21] and other authors only use "*Gegensatz*" (contraries). In English, depending on the dialectic statement being considered, the two words are usually employed.

qualifiers possess opposite aspects. Sexual intercourse can be an act of love or hate; the whore is simultaneously despised and desired; insult or praise are two sides of the same attitude; positive situations (such as hurrying up) can also be negative. Natural languages make spontaneous use of this unity of opposites, an aspect which can be seen throughout the centuries and across languages.

## Unit and struggle of opposites

*Adversative* conjunctions pose a formidable logical challenge. It is frequent to interpret adversative conjunctions as variations of the logical function **AND**. According to this, an expression of the type:

> $a$ **but** $b$

is usually interpreted as $a$ **AND** $b$ with the added element that, within the statement, *the presence of* $b$ *must be taken especially into account*. It is worth noting that this is the reason why there is a certain asymmetry in the role of the two elements, $a$ and $b$. In many cases, adversative conjunctions are usually interpreted in this manner, but their use does not end with this. We will present some examples to introduce new situations. Let us consider the following:

> Those who love, hate [20]

In this case, it is established that love is inseparable from hate, but there is no doubt that these two statements are: "those who love, [also, **but**] hate" and "those who hate, [also, **but**] love". The order seems indifferent although they state the same thing.

The possibility of constructing statements capable of being interpreted in two different ways is another use of the conjunction **but**. In the following joke, quoted by Sigmund Freud (1856, 1939), it is used along with another function:

> Serenissimus asked a stranger by whose similarity to his own person he had been struck:

[20] The title of a novel by Adolfo Bioy Casares and Silvina Ocampo (1946).

> –Was your mother in the Palace at one time?
>
> And the repartee was:
>
> –No, **but** my father was. [27]

In this case, the conjunction **but** has a very special role. This fragment contains two possible interpretations, which are indicated by the conjunction. It is possible to interpret that the similarity responds to mere chance, and it is also possible to interpret, against the ruler's suggestion, that his father–and not his mother–is responsible for the resemblance. We understand, and this will be reinforced by other examples, that the conjunction **but** expresses a different logical function. This statement, as are many jokes and word games, is the intellectual equivalent of Louis A. Necker's (1786, 1861) cube:[21] a double interpretation is present, and we cannot decide which of the two possible interpretations the speaker is referring to.

A second example by Freud shows yet another use of the conjunction **but**:

> Frederick the Great heard of a preacher in Silesia who had the reputation of being in contact with spirits. He sent for the man and received him with the question
>
> –You can conjure up spirits?
>
> The reply was:
>
> –At your Majesty's command. **But** they don't come. [27]

In this example, the result is also a quip, but one of a different logical nature. Here, instead of two possible interpretations, a contradiction occurs. The reply, in very plain terms, intends to say:

> I can conjure up spirits **but** they do not come.
>
> I can conjure up spirits **but** I cannot conjure up spirits.

[21] It refers to the perspective image of a transparent cube which can be interpreted as being seen from behind as well as from the front. In a similar way, it is possible to draw a staircase that can be seen as from above or below. There are many examples of figures containing double, and even triple, interpretation.

This second statement is the most precise (but also does away with the joke). The conjunction **but** allows us to construct *a contradiction which has the value of a joke*. It is also capable of enabling double interpretation, as in the first example: at the same time, it allows us to say that spirits can be conjured up unsuccesfully. This example is symmetrical. It is the same thing as saying "spirits do not come **but** I can conjure them up".

No one expresses this function better than Sister Juana Ines de la Cruz (1651, 1695) in an exceptional poem:[22]

| | |
|---|---|
| *En dos partes dividida* | In two parts divided |
| *tengo el alma con confusión:* | is my soul in confusion: |
| *una, esclava de la pasión,* | one, a slave to passion, |
| *y otra, a la razón medida.* | and the other, tied to reason. |
| *Guerra civil, encendida,* | Civil war, lit up, |
| *aflige el pecho importuna:* | intrudingly torments the soul: |
| *quiere vencer cada una,* | each aspires to triumph, |
| *y entre fortunas varias,* | and among various fortunes, |
| *morirán ambas contrarias* | they will both die opposed |
| *pero vencerá ninguna.* | but none shall prevail. |

The poem expresses the relationship between passion and reason as a unity and struggle of opposites, which is similar to the constructions of adversative conjunctions in everyday language.

## The non-logic of love

Defining love has been a concern for practically all poets. There is a common element to many of them: describing the state of love as something contradictory, difficult to express. This idea will appear time and again throughout the centuries, at least for as long as written texts have been preserved.

The first to describe this notion was possibly the Ionian Anakreon (-572?, -485?), in the verse attributed to him:

[22] This fragment is a stanza of "*Dime vencedor rapaz*" (Tell me rapacious conqueror).

*I both love and do not love; and am mad and not mad.*[23]

Although this case may be somewhat ambivalent, the poetry of Gaius Catullus (-84?, -54) poses no questions as to the contradictory feelings sparked by love:

*Odi et amo. Quare id faciam fortasse requiris?*
*Nescio, sed fieri sentio et excrucior.* [9, Carmen #85][24]

Many centuries later, a fragment of the famous poem by Neruda introduces the same idea:

Ya no la quiero, es cierto, **pero** tal vez la quiero. [25]

It is a question of interpreting meaning from a logical standpoint, since there is no doubt that, until now, no one has ever been concerned with the "non-logic" of this text. Also, practically everyone will agree that the sentence conveys a confusing blend of feelings which–nonetheless– is easily interpreted in a spontaneous manner. This verse indicates that it is equally valid to state "I love her" and "I do not love her'.

If we were only to resort to binary logical functions we would be at a loss. The statement

I do *not* love her **OR** I love her

poses no challenge because it is universally valid regardless of the author's feelings. It is clear, then, that in order to express doubt, two opposing feelings coexist. It would be more appropriate to say:

I do *not* love her **AND** I love her.

[23] This text is cited in many places, but it does not appear in John Addison's edition, London, 1735 (Google Books), or in Alexandre Marchard's, Paris, 1884 (Gutenberg Library).

[24] I hate and love. Why do I do this, perhaps you will ask? I do not know, but I feel it happens and it tortures me.

[25] I no longer love her, that's certain, **but** maybe I love her. Verse from poem 20, *Twenty love poems and a song of despair*, Pablo Neruda.

This statement is *universally false*. This is why the conjunction **but** is used, since it allows to establish a material contradiction with a new meaning. The paradoxical statement appears as something halfway between the logical functions **AND** and **OR**, and for this reason a different conjunction is used. In a strict sense, in this function, **but** is at an equal distance from both. It is not true–as linguists are usually quick to point out–that **but** is a *modified* **AND**: it is an entirely new logical function.

Francesco Petrarch (1304, 1374) wrote a sonnet–which has been translated and plagiarized over and over in many languages[26]–containing a classical definition of love through the use of pairs of contradicting elements.

The importance of this sonnet resides in its peculiar structure: a succession of contradicting pairs joined by the conjunction **AND**. The passage from a simple contradiction to a set of contradictions joined by a conjunction is a new structure which is of great importance to dialectic logic. This justifies including the entire sonnet here. At the same time, the number of translations–which fail to mention Petrarch–additionally reaffirms the importance of the new structure discovered by the poet.[27]

| | |
|---|---|
| *Pace non trovo, et non ò da far guerra,* | I find no peace, and all my war is done; |
| *e temo e spero, e ardo e sono un ghiaccio,* | I fear and hope, I burn, and freeze like ice; |
| *et volo sopra 'l cielo e giaccio in terra,* | I fly aloft, yet can I not arise; |
| *e nulla stringo e tutto 'l mondo abbraccio.* | And nought I have, and all the world I seize on. |
| | |
| *Tal m'à in pregion, che non m'apre né serra,* | That locks nor loseth, holdeth me in prison, |
| *né per suo mi riten né scioglie il laccio,* | And holds me not, yet can I scape no wise, |
| *e non m'ancide Amore, et non mi sferra,* | Nor letteth me live, nor die, at my devise, |
| *né mi vuol vivo, né mi trae d'impaccio.* | And yet of death it giveth me occasion. |

[26] Olivier de Magny (1529, 1561) translated it into French in the sonnet which begins with: "*Je cherche paix, et ne trove que guerre*"; Pierre de Ronsard (1524, 1585) has also imitated him ("*J'espère et crains, je me tais et supplie*"), Luise Labé (1524, 1566), see *Première Anthologie Vivante de la Poésie du Passé*, Paris, 1951. In Portuguese there is the version by José Bonifácio de Andrada e Silva (1827, 1886), Frei José de Salamanca, see *Os mais belos sonetos que o amor inspirou*, Rio de Janeiro, 1965.

[27] The Modern version used here is from Thomas Wyatt (1503, 1542)–who introduced the sonnet in England–and translates it in a sonnet entitled "*Description of the Contrarious Passions in a Lover*", see *The Penguin Book of Sonnets*.

| | |
|---|---|
| *Veggio senza occhi e non ò lingua et grido,* | Without eye I see; without tongue I plain: |
| *et bramo di perir e chieggio aita,* | I wish to perish, yet I ask for health; |
| *et ò in odio me stesso, et amo altrui.* | I love another, and I hate myself; |
| | |
| *Pascomi di dolor, piangendo rido,* | I feed me in sorrow, and laugh in all my pain, |
| *egualmente mi spiace morte e vita:* | Lo, thus displeaseth me both death and life, |
| *in questo stato son, Donna, per vui.* | And my delight is causer of this strife. |
| [74, Le Rime, CXXXIV] | |

Petrarch–and his imitators–employed pairs of contradicting ideas joined by the conjunction **AND**, which establishes the contradiction. At the same time, the different pairs are joined by commas which replace a certain undefined logical function which can be both **AND** and **OR**. In line with the previously analyzed, the logical function expressed by the conjunction **but** can also be considered.

It is worth mentioning here a well-known sonnet by Lope de Vega (1562, 1653), which also attempts to define love. Not known for ever surprising anyone in its non-logic, the text is admirable in its simplicity:[28]

| | |
|---|---|
| *Desmayarse, atreverse, estar furioso,* | To faint, to dare, to be enraged, |
| *áspero, tierno, liberal, esquivo,* | coarse, tender, liberal, elusive, |
| *alentado, mortal, difunto, vivo,* | encouraged, mortal, dead, alive, |
| *leal, traidor, cobarde y animoso;* | loyal, traitor, coward **AND** brave; |
| [ . . . ] | |
| *esto es amor, quien lo probó lo sabe.* | this is love, he who has had a taste of it so knows. |

The definition elaborated by Lope is made up of a long list of contradictory elements–without intending to assume anything about the meaning of "contradictory"–, *separated by commas.* The author uses commas because it is not easy to write the conjunction–or conjunctions–

[28] A sonnet by Lope de la Vega cited in a multitude of anthologies, it possibly belongs to a theater play. Translated from a version included in "Clásicos Castellanos", Ediciones de "La Lectura", Madrid, Volume I, Lope de Vega, Sonnet CXXVI.

which bind the entire set together. Unlike Petrarch, only in one verse of the sonnet does Lope write the conjunction **AND**.

Punctuation signs manage to make up very clear pairs of opposing elements. The authors' intention is to make up a list of contradictions that characterize loving passion. In fact, the technique of the paradox is repeatedly used and the comma–or the **AND** conjunction–aids in expressing the manner in which these contradictions come together. It is interesting to note that, with the exception of a certain possible asymmetry, Lope's statements could be written as:

faint $*$ dare

loyal $*$ traitor

coward $*$ brave

and so forth. This notation tries to show that there is a clear link between the use of a conjunction, represented by $*$, which has been replaced by a comma in the sonnet. However, we do not intend to find, at least for the time being, the conjunction or logical function that is replaced by *the commas which bind the contradictions together*. This problem will be clarified further ahead. For now, we can only accept that this logical function is feasibly an associative and commutative operation, as called for by the interpretation of the definition attempted by the sonnet.

The examples shown allow us to assume that there are more logical structures in the brain than those considered by Russell in mathematics. This will be our topic of study.

## Becoming

Unlike the penetration of opposites–which has no precise expression in the natural languages–the notion of "becoming" does: the verb to *become*.[29] Bearing this in mind, we can begin to analyze it. Some literary examples will assist us as we move forward.

[29] The verb comes from the Latin *devenir*, to come, to arrive. The German language has the verb *werden*, which is the auxiliary of the future, and also the verb *devenir*; the English languages uses "becoming", which is related to the Dutch *bekomen*, a Germanic term.

Jesus's preaching as gathered in the Gospels contains many examples of "becoming" statements. We will start by John the Baptist's preaching as he announces the arrival of Jesus:

> Every valley shall be filled, and every mountain and hill shall be brought low; and the crooked shall be made straight, and the rough ways shall be made smooth. [Lk 3:5][30]

This passage contains four "becoming" statements where it is announced that each of the natural elements will become its opposite. The beatitudes also contain a logical structure made up of opposites, although it is somewhat more complex:[31]

> Blessed be ye poor: for yours is the kingdom of God. Blessed are ye that hunger now: for ye shall be filled. Blessed are ye that weep now: for ye shall laugh [ . . . ] Woe unto you that are full! for ye shall hunger. Woe unto you that laugh now! for ye shall mourn and weep. [Lk 6:20–25]

In this case there are some opposing "becoming" elements:

> hunger **becomes** satiety **but** satiety **becomes** hunger
>
> weeping **becomes** laughter **but** laughter **becomes** weeping

Here, each human situation becomes its opposite, and *reciprocally*: laughter **becomes** weeping **becomes** laughter and so on. This structure repeats itself in other passages:

> And, behold, there are last which shall be first, and there are first which shall be last. [Lk 13:30 and also Mt 19:30, Mt 20:16, Mk 10:31]
>
> For whosoever exalteth himself shall be abased; and he that humbleth himself shall be exalted. [Lk 14:11 and also Lk 18:14, Mt 20:27, Mt 23:12]

[30] All references from the Bible are taken from [3].

[31] The Gospel of Luke contains the oldest version of the account. In this way, for example, Matthew's beatitudes do not have the logical precision of Luke.

In the *Romancero del Cid*–anonymous Spanish poems from the $11^{th}$ century–we find *Doña* Jimena who once again seeks justice from the king for the Cid killing her father, a story which is recounted in the poem:[32]

| | |
|---|---|
| [ . . . ] *al que mi padre mató* | [ . . . ] the man my father killed |
| *dámelo para casar,* | give him to me to marry, |
| *que quien tanto mal me hizo* | for whomever inflicted such deep hurt |
| *sé que algún bien me fará.* | I know some good will do me at last. |

A clear example of becoming through the opposite appears here:

inflicting such deep hurt **becomes** some good.

In many of his poems, Heinrich Heine (1797, 1856) proposes a form of dialectic logic, see [38]. The following poem presents a case of "becoming":

| | |
|---|---|
| *Es liegt der heisse Sommer* | Warm summer |
| *Auf deinen Wängelein*; | dwells upon thy cheeks, |
| *Es liegt der Winter, der kalte*, | Cold, frosty winter lies |
| *In deinem Herzenchen klein.* | in thy little heart, fair child. |
| | |
| *Das wird sich bei anders,* | Yet these, I think, as years grow on, |
| *Du Vielgeliebte mein!* | Will play a different part; |
| *Der Winter wir auf den Wangen,* | Then, winter on thy cheeks shall be, |
| *Der Sommer in Herzen sein.* | And summer in thy heart. |
| [43, *Lyrisches Intermezzo*, 48] | |

Heine proposes a double transformation:

summer in cheeks **becomes** winter in cheeks

winter in heart **becomes** summer in heart

[32] Own translation.

The becoming of life will invert the face and the heart to their opposites. In two passages of his work *Tristan und Isolde*, Richard Wagner (1813, 1883) suggests processes related to the becoming of love:

| | |
|---|---|
| *Des Welternwerdens* | of the **becoming** of the world |
| *Walterin . . .* | regulating . . . |
| *in Liebe Wandelnd den Neid* | you turn envy to love |
| [95, II, 1] | |

The verb to become is explicitly used here (*werden*) to state: envy **becomes** love. In the following scene we find this statement made by Tristan, followed by the same statement by Isolde:

| | |
|---|---|
| *Tristan du* | you Tristan |
| *ich Isolde* | I Isolde |
| *nicht mehr Tristan* | never more Tristan |
| [95, II, 2] | |

This case deals with the transfiguration operated by love:

Tristan **becomes** Isolde    Isolde **becomes** Tristan

The capitalist society, and money in particular, is the object of several "becoming" statements. We will begin with the famous poem by Francisco de Quevedo (1580, 1645) on the power of money (*La pobreza, el dinero* [Poverty, money]):[33]

| | |
|---|---|
| *¿Quién hace al tuerto galán* | Who can turn into a gallant a one-eyed man |
| *Y prudente al sin consejo?* | And make a prudent gent out of a dunce? |
| . . . | |
| *¿Quién hace de piedras pan,* | Who can turn stones to bread |
| *Sin ser el Dios verdadero?* | And is not the One True God? |
| *El Dinero.* | Money. |
| | |
| *¿Quién la Montaña derriba* | Who brings together mountains and valleys |

[33] Own translation.

| | |
|---|---|
| *Al Valle; la Hermosa al feo?* | The Pretty Girl and the Ugly Man? |
| … | |
| *¿Y quién lo de abajo arriba* | Who could easily raise up |
| *Vuelve en el mundo ligero?* | what was once downcast? |
| *El Dinero.* | Money. |

Money is defined by means of a succession of becoming statements such as: one-eyed man **becomes** gallant, stones **become** bread, and others.

In *Das Kapital*, Marx also provides a direct example of becoming to contribute an explanation for the capitalist commercial cycle.

> *In der Zirkulation $W - G - W$ hat also die Verausgabung des Geldes nichts mit seinem Rückfluß zu schaffen. In $G - W - G$ dagegen ist der Rückfluß des Geldes durch die Art seiner Verausgabung selbst bedingt. Ohne diesen Rückfluß ist die Operation mißglückt oder der Prozeß unterbrochen und noch nicht fertig, weil seine zweite Phase, der den Kauf ergänzende und abschließende Verkauf, fehlt.* [ … ] *Form dieses Prozesses ist daher $G - W - G'$, wo $G' = G + \Delta G$, d.h. gleich der ursprünglich vorgeschossenen Geldsumme plus einem Inkrement.*[34] [60, I, 4, 1]

By using the notation $\to$ for becoming, as suggested by the text, the following is expressed:

$$\cdots\ W \to G \to W' \to G' \to \cdots$$

It is a cycle with no beginning and no end, whose rotation increases both money, $G$ (*Geld*), as well as merchandise, $W$ (*Ware*), in quantity.

[34] In the circulation $W - G - W$, the investment of money has nothing to do with its return. On the other hand, in $G - W - G$, the return of money depends on the nature of the investment. Without return, the operation would fail and the process would be interrupted since it is lacking its second phase, which is complementary and is its final state. [ … ] The shape of this process is, then, $G - W - G'$, where $G' = G + \Delta G$. That is, it is equal to the originally invested sum plus an increment of money. [63]

Becoming through the opposite is the stabilizing mechanism of some complex systems. The structure of "becoming" is a way to resolve the contradiction between two tendencies which must coexist, opposite to one another and dependent upon one another. In nature, the balance between predators and preys is stable, yet fluctuating: there are times when the prey population increases, leading to an increase in predators and so forth. In a capitalist society we can observe an alternation between opposite parties in power: the so-called *right wing parties*–which favor companies and businesspeople–and the so-called *left wing parties*, which support the social conquests of the working class. An alternation between popes can be seen in the Roman Catholic Church: a theologian pope–concerned with doctrine–is succeeded by a pastor pope, who is more akin to strengthening the faith of his following.

## Argumentation in sciences

Argumentation is a process of reasoning which does not readily respond to binary logic. Let us consider an emblematic case of argumentation: the courtroom. An accuser offers his or her arguments while a defender presents theirs. Each one attempts to refute the other, as a judge or jury must come to a decision. Which argument better adjusts to the truth? The mere action of posing this question escapes binary logic.

In a process of argumentation, all the parties hold truthful affirmations. However, the judge needs to make an assessment. Sometimes, he will employ the concept of "more truthful than", which exists in human thought and modal logic but is missing from binary logic.

The same thing happens with the sciences. Scientific theories must be supported in order to convince the scientific community, with the occasional dispute between an old, accepted theory, and a new one that challenges it. The scientific community must then evaluate using the "more truthful than" criterion.

In order to illustrate the problem using an actual example from history, we will analyze the argumentation presented by Isaac Newton (1642, 1727) on universal gravitation–undoubtedly, one of the clearest examples of argumentation in science. Newton not only revolutionized mathematics, mechanics and optics, but also epistemology. We will not worry here about the laws of motion, but simply consider his argumentation on gravitation.

Newton was aware that this was a leap in an entirely new direction for the scientific methodology supporting gravitation, that he felt compelled to add a note at the end of the third edition of the book to clarify this point. The famous passage of the *Principia* "*hypothesis non fingo*" (I do not feign a hypothesis) can be found there.

> *Rationem vero harum gravitatis proprietatum ex phænomenis nondum potui deducere, & hypotheses non fingo.*[35] [68, III, *Scholium Generale*]

The introduction to the "System of the World", book III, presents a difference between its first (1687) and third edition (1728). In [67, III] the author begins with a set of 9 *hypotheses*. The third edition [68, III] however, opens with 4 *methodological rules* and 6 *experimental phenomena*. doubt, an epistemological shift in Newton's way of thinking, despite the fact that the actual *content* is practically the same in the two versions.[36]

[35] "I have not as yet been able to discover the reason for these properties of gravity from phenomena, and I do not feign hypotheses." The translation used here is from [70], p. 943.

[36] For the sake of abbreviation, I will call the hypotheses H, the rules R, and F will stand for phenomena. H1 is equivalent to R1 and establishes Occam's principle: do not multiply the causes beyond necessity. H2 is equivalent to R2 and establishes that equal effects must obey to equal causes. H3 is equivalent to R2, but in R2, Newton feels compelled to give a lengthy explanation which is missing from H3. This rule establishes that the experimental properties of bodies related to quantity are universal, such as gravity. R4 establishes the *methodology of induction*, except if cases are found where does not hold true, which forces to change it. This is a precursor of Popper's refutation. H4 establishes that the Solar System is at rest, while this statement has disappeared from the third edition. H5 is equivalent to F1 and establishes the third law of Kepler for Jupiter's satellites. F2 establishes that Saturn's satellites meet the third law of Kepler, as discovered by Cassini between 1671 and 1684, unknown at the time of the first edition. H6 is equivalent to F3 and establishes that the 5 planets orbit around the Sun. H7 is equivalent to F4 and establishes the third law of Kepler for the 5 planets. In both cases he presents the figures, but F4 has more information. H8 is equivalent to

This shift from "hypotheses" to "methodological rules" and "experimental phenomena" establishes the birth of modern science. It is not about enunciating axioms–hypotheses, to use the terminology of the first edition–but to having a basis on actual experiments or observations. Let us examine these "experimental phenomena".[37]

The following are the astronomical observations explicitly mentioned by Newton:

- The Copernican description of the Solar System: planets orbiting around the Sun.
- The first law (K1) of planetary motion by Johannes Kepler (1571, 1630), which establishes that the areas swept by the line that joins together the "central" and the "moving" body, are equal during an equal interval of time.[38]
- The third law of Kepler (K3), which establishes that the period times of the orbits of any two planets is the ratio of the $3/2^{th}$ power of the mean distance from the "central" to the "moving" body. This law was observed on three occasions: for the entire Solar System, for Jupiter and for Saturn.

Starting at K1, through theorems [67, 68, I, *Theorema* i, ii, ii], Newton proves that the force which determines the orbits which comply with this law are forces directed from the "moving" body to the "central" body, see [67, 68, III, *Theorema* i, ii].[39] As from K3, through [67, 68, I, *Theorema* iv, *Corol.* vi], he will show that the forces are in inverse proportion to the square of the distance between the two bodies.[40]

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F5 and establishes the first law of Kepler for the 5 planets. H9 is equivalent to F6 and establishes the first law of Kepler for the Moon.

[37] Euclid[23, I] had already made the distinction between three types of statements: the 23 definitions, the 5 postulates (axioms) and the 5 common notions (basic rules of logic).

[38] Kepler derived this law for Mars from Tycho Brahe's (1546, 1601) records, which was accepted by Newton for the Moon since its motion is practically circular.

[39] The geometric simplicity of the demonstration of these theorems is simply astounding. There is no doubt that this was the main idea behind the theory of gravitation.

[40] Theorem IV shows that the centripetal forces of *circular motion* are proportional

It is clear that in demonstrating the law of inverse proportion to the square of the distance, Newton's reasoning used circular motion, since the eccentricities of the ellipses of the actual orbits are very small. However, once the Law of Gravitation was established, he laboriously reconstructed the elliptical and parabolic movements of the Solar System.

But there was one very important element missing from all of this: the motion of the Moon. He will address this in [67, 68, III, *Theorema* iii, iv]. Here, Newton *simply* verifies that the centripetal acceleration of the Moon and bodies falling near the surface of the Earth also comply with the inverse square-law.[41]

Newton's argumentation was overwhelming: the uniform circular motion conformed to K1; the Solar System, Jupiter's, Saturn's and the Earth's satellites complied with K3. He finished off his argumentation with a theoretical demonstration of Kepler's Second Law (K2), which established that planetary motion is elliptical, with the Sun in one of the foci. One final confirmation was the reconstruction of a comet discovered by John Flamsteed (1646, 1719). The *Principia* banished

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to the squares of the arcs described within a unit of time, divided by the radius. In modern language, the acceleration in a circular motion is $\alpha = v^2/R$. Corollary VI establishes that if the periods of a circular motion are proportional to the $3/2^{th}$ power of the radius and the speeds are inversely proportional to the square root of the radius, then, the centripetal forces are inversely proportional to the square of the radius. In modern language, the period of a circular movement is $T = 2\pi R/v$, then after $T = k\,R^{3/2}$ it results in $v = 2\pi/k\,R^{1/2}$ and then $\alpha = K/R^2$, where $K = (2\pi/k)^2$ is a constant of proportionality.

[41] The measurements made by several astronomers show that the distance from the Moon to the Earth is 60 times the radius of the Earth; its period T is 27 days, 7 hours, 43 minutes, –T = 39,343 minutes– and the Earth's circumference is $c = 2\pi\,r = 123.2496 \times 10^6$ feet of Paris (these are the figures employed by Newton). Therefore, the centripetal acceleration of the Moon is $\alpha = 4\pi^2 \times 60\,r/T^2 = 120\,\pi\,c/T^2 = 120\,\pi \times 123.2496 \times 10^6/(39,343)^2 \times 10^6 = 120\,\pi \times 123.2496/(39343)^2 = 30.0$ feet/min$^2$. Then, the "fall" of the Moon towards the Earth *in one minute* is $30.0\,t^2/2 = 30.0/2 = 15.0$ feet. Newton calculates 15 1/12, that is 15.08. The acceleration on the Earth's surface, according to the law of gravitation, would be $60 \times 60$ times greater, that is, 30,0 feet/seg$^2$, and from this, the fall *in one second* would also be 15.0 feet. A foot of Paris is 326.6 millimeters. Therefore, the acceleration of gravity on the surface of the Earth, as per Newton's calculus, is $30.0 \times 0.3266 \approx 9.8$ m/seg$^2$ which matches current measurements.

the old planetary theory proposed by Claudius Ptolemy (100?, 170?). However, there was no direct experimental evidence on gravitation.

Then, what logical value did universal gravitation have? It was undoubtedly a "truthful" statement, since it related to "truthful" affirmations of mathematics and geometry and to experimental observations. However, to the eyes of the $20^{th}$ century and our own, there is no doubt that this was a "temporary truth", given that the idea of universal gravitation was outdone by the notion of a "curved space" as defined in Albert Einstein's General Relativity (1879, 1955). Ultimately, no scientific affirmation, no matter how consolidated and accepted it may be, is an "absolute and final truth"–while it is simply more truthful than the previous theory, it is surely less so than any other more elaborate theory which might be developed in the future. We must once again turn to the notion of "more truthful than".

In 1798, Henry Cavendish (1731, 1810) completed an experiment designed by geologist John Michell: the direct measurement of the attraction between fixed, 350-pound lead spheres, and 1.6-pound, mobile spheres, see [10]. The measurement was taken using the deviation of a torsion scale, carefully insulated to avoid problems with temperature, electricity, magnetism and air currents. Thus, he found the value of the gravitation constant $G = 6.754 \times 10^{-11}$ Newton m$^2$/kg$^2$, which is quite similar to the currently accepted value.[42]

## Synchronic and diachronic opposites

Not all pairs of elements are opposing elements.[43] The idea of opposites applies to two cases that have a different temporal relation, even if classical authors have never noticed this difference.[44] Los contrarios

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[42] It is interesting to calculate the attraction between these lead spheres. The masses weight 160 and 0.7 kilos approximately and are placed roughly 20 cm apart. The force of attraction is $G \times 160 \times 0.7/0.2^2 \approx 1.9 \times 10^{-7}$ Newton, of the order of a small fraction of a microgram.

[43] Mao Zedong (1893, 1976) [59] carries out a detailed analysis of the notion of dialectic opposites.

[44] Ferdinand de Saussure (1857, 1913) in his *Cours de Linguistique Générale* (1906–1911) [88], was one of the first scholars–of the human sciences–to notice this major conceptual difference.

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- Two simultaneous and complementary aspects of the same reality, which we will refer to as *synchronic opposites*.
- Two successive and opposed aspects of the same reality, which we will refer to as *diachronic opposites*.

We can illustrate this idea with some simple examples. In the realm of artistic creation, we are able to identify two roles we do not hesitate to call *opposites*: the role of the creators and that of the critics. The question is then, which type of opposite do creators and critics represent? In reality, while they may belong to any of the two types, the two give way to very different situations. The reasonable thing would be to assume that art critics and creators are *synchronic* opposites: two different groups of people who are simultaneous, complementary and concerned with the same reality. However, they may also constitute *diachronic* opposites: in this case, each critic can also be a creator who alternatively performs the two activities. But this may lead to the undesirable situation of an activity which becomes corrupted.[45]

It is possible to find a variety of synchronic opposites. The following is a brief list:

- For every right consecrated by the law–either explicitly or implicitly–there is an opposite right which limits its scope, such as, for instance, freedom of expression and defense of honor.
- For classical Socialists, opposed social classes have opposing interests and one cannot exist without the other: a factory owner cannot exist without workers and vice versa.

[45] The mechanism of corruption is simple and very widespread in Latin America. $A$, acting as a critic, says that $B$ is an exceptional artist. In turn, $B$, in his role as critic, returns the favor by saying that $A$ is also an exceptional artist. If critics and creators become diachronic opposites, the result is usually the corruption of the arts and critics. Something similar occurs in science. Publications are accepted or rejected by reviewers through a peer review process, which poses the same dilemma: the role of the author and that of the evaluator become mixed up, giving way to a mechanism which is often perverted.

- After Freud,[46] love and hate are considered inextricable opposites–one cannot exist without the other.
- Beauty and ugliness are yet another example of inextricable opposites. One cannot exist without the other, in permanent unity.[47]
- Oscar Wilde extends the idea of opposites by stating that every artistic manifestation partakes in a contradiction of this type: the opposite of a valid artistic current is also a valid artistic current.[48]
- The universal and the specific are classical opposites formulated by Leo Tolstoy (1826, 1910) as inextricable[49] and have been analyzed by philosophers since ancient times.
- In philosophy, materialism and idealism are two opposing, dual currents. With regards to politics, the followers of Locke (liberals) and Rousseau (social reformists) are usually identified as opposing and dual currents.
- Quantum mechanics and relativistic mechanics are two opposing, complementary theories, both simultaneously accepted by the scientific community to explain different aspects of the universe.

Diachronic opposites appear in every analysis of the origin and movement of phenomena.[50] What follows is a brief list of classical examples:

- Which came first, the chicken or the egg? This classic conundrum evidences the existence of diachronic opposites. The egg

[46] Prior to Freud, there are myriad examples in poetry in which this identity and struggle is established, as was introduced in the previous chapter. The *odi et amo* of Roman poet Catullus (–84?, –54?), was one of the first examples of this.
[47] The issue of beauty has occupied poets and philosophers, with no straightforward solution. The difficulty lies in this inextricable contradiction. In *Les Fleurs du Mal*, Charles Baudelaire (1821, 1687) was one of the first to display this identity.
[48] This idea is also necessary in politics, even if it has not been formally stated or explicitly accepted. This statement has interesting consequences.
[49] A quote–which I have not been able to precisely locate–attributed to Tolstoy, formulates these opposites: *speak of your town and you will speak of the world*.
[50] A lattice of the type **rD**$\infty$ is applicable in this case, although we will not delve into the details of this lattice in this study.

engenders the chicken and the chicken engenders the egg. This problem is poorly formulated, as Darwin showed in his analysis of the evolution of the species.

- The evolution of the human hand and brain is another case: one gives way to the other in a successive and repeated manner.
- Natural selection presents itself as the succession of a mutation followed by the survival of the fittest. This process acts in a reiterated, endless manner. The two previous cases are examples of this idea.
- As argued by historical materialism, human societies succeed themselves by opposing one another.

The fact that the hand and the brain coexist in time does not change their character: they are diachronic, and not synchronic, opposites.[51] Synchronic opposites have no diachronic link: struggling classes are all outdone by a new society.[52]

Aristotle was the first philosopher to discover that the only alternative to the existence of diachronic opposites was to accept the existence of an external, static God, who is responsible for motion. Generally speaking, there are only two ways of understanding motion: like Aristotle does–God created the egg and the chicken simultaneously (as synchronic opposites)–, or by the action of diachronic opposites, in the manner of Darwin, arguing that this process accumulates quantitative changes which result in a qualitative change with the appearance (or disappearance) of new species.

Accepting the existence of dialectic opposites which are necessary for understanding the notion of becoming is a very important methodological principle which will be used in numerous occasions. Con-

[51] To consider that the hand and the brain are synchronic opposites leads to arguing for the separation between manual and intellectual work and other notions which may lead to false interpretations. This example warns of the risks of confusing the two types of opposites.

[52] When the *Communist Manifesto* [61] states that the unity and struggle of classes in capitalist society–the bourgeoisie and the proletariat–will be outdone by the supremacy of the proletariat, it turns synchronic into diachronic opposites and makes an important mistake which it is not relevant to analyze here.

versely, not accepting opposites leads to an oversimplification and a schematization of the studied reality.[53] In the studies contained in this book there will be many examples of opposites which are at the basis of historical becoming.

## Isomorphisms and Homomorphisms

Statements can be made in different languages. One can assume that all statements worthy of consideration can be translated from one language into another: that language should not be an impediment in expressing knowledge.

The operation of translating a universe of statements in Spanish into a universe of English counterparts, if everything happens as it should, is an *isomorphism*.[54] The logical structures are preserved and everything glides along smoothly. In practice, the situation is not that simple and encountering difficulties is not rare, especially with regards to *temporal* relations. Let us take a look at some examples:

| | |
|---|---|
| $x$ **es** enfermo | $x$ **is** sick |
| $x$ **está** enfermo | $x$ **is now** sick |
| $x$ **deviene** enfermo | $x$ **become** sick |

The correspondence between these statements in the different languages is not always straightforward. The Spanish language allows us to clearly distinguish between statements, which is infrequent in other languages. Regardless of this, we can accept that the change from one language to another is no more than an *isomorphism*–a perfect correspondence–with no major consequences for the science of logic.

Classical logic consists in associating–or projecting–each statement

[53] A classic example can be found in Jesus' statement: *whoever is not with me is against me*. Although there are different wordings for this text, it can be found in the three gospels: Mt 12:30; Mk 9:40 and Lk 9:50. There are many different levels between being in favor or against something, when that which is being debated represents synchronic opposites. To not accept this idea is an oversimplification of reality, and in this case, becomes the seed for religious intolerance.

[54] The word isomorphism–as do homomorphism and lattice–has a precise meaning. The present and the following chapter informally introduce these notions which will be defined more precisely moving forward.

over a structure formed by two values: "true" or "false". As we acknowledge that the natural language contains more complex structures than mere binary logic, the structure over which the statements are associated or projected becomes more complex. This mathematical structure can be identified by means of a *lattice*. We will not delve in mathematical notions in these initial chapters, but rather consider it as a useful structure or diagram for representing ideas.

The issue of the homomorphism is slightly more important within the universe of statements. As we have mentioned before–with increasing precision–, a logical structure is the result of a "correspondence", a "projection" or an "understanding" of the universe of natural statements over an abstract structure. This operation is a *homomorphism* which, due to its characteristics, we must consider as a *universal homomorphism*. We will come back to this subject many times.

## Metaphysics and dialectics

The metaphysical interpretation of the universe consists in establishing a homomorphism between reality and a structure chosen in a more or less whimsical manner. The planets, the strings of a musical instrument, the cardinal points, the elements, everything is possible. Here, it is metaphysical thought which prevails, and not the scientific study of reality.

Logic stems from the idea that this homomorphism must be, in some way, irreducible, with no new interpretations, final, in a word.

If we assume that all interpretation must be a logical interpretation of the universe, we must accept that the elementary operations **AND** and **OR** are preserved, which would make the homomorphism lead to a lattice. Thus, the simple idea of logic is born as we conceive of it today, as the formal image of the universe.

Logic is, ultimately, the result of a homomorphism that preserves the structural properties of knowledge.

# Natural dialectics

## Logic and dialectics

An experimental study of human logic reveals an exceptionally ample panorama. Even leaving developmental "genetic" changes of human thought or forms which may be typified as "pathological" aside, there is still a vast field to explore. This field extends from the forms used by philosophers in the past–the creators of the Upanishads, Lao-Tze, the pre-Socratics, especially Heraclitus, and others–to the dialectics of Georg W. F. Hegel (1770, 1831), Friedrich Engels (1820, 1895) and Karl Marx (1818, 1883) in modern times. Generally speaking, we will refer to dialectics as the entire formalized and rationalized thought of human beings.

Logic, from Aristotle to our days, comes to us as something natural. Cultural tradition and education influence this fact, but, above all, *it is natural because it has been imposed on the human brain by the evolution of the species*.

If we wanted to support the validity of Aristotle's logic, we could make four powerful arguments to reinforce its natural character and universal application in science.

The first argument is historical. The Euclid's *Elements*, written 22 centuries ago, shows us that logical structures have not changed, at least for the last thousands of years. The historical continuity of formal thought, which was lost in classical Egypt, is a first and foremost argument.

Modern languages are able to express any logical Boolean structure. *This fact has occurred without the intervention of academic scholars*. It is a natural fact and constitutes the second, formidable argument which has been analyzed in previous sections.

There is still little knowledge on how the human brain works precisely. However, from what we do know, *neuronal connections have been* *found that establish elementary binary logical circuits*, which is also a natural fact. This would be the third argument

The fourth argument is scientific in nature. In history, the astronomy of agricultural calendars made widespread use of mathematics. Thanks to Euclid and other Alexandrian scientists, geometry became a branch of deductive mathematics. With Galileo and Newton, physics became a mathematical science. After Lavoisier, chemistry suffered the same fate. In the present century, thanks to molecular genetics, biology is also following in the footsteps of formalization. This process shows us that the fundamental tool for analyzing matter is logic, which is an astounding argument in itself.

In order to study dialectic logic, we must walk a similar path. We need to look for dialectics whenever we find an area which cannot be analyzed in traditional logical terms–look for it in the cracks of binary logical thought. For this reason, the sources of dialectics can be traced back to those of traditional logic.

In this chapter we will show that dialectics is a *natural* activity of human thought. If this is so, dialectics must then be applicable to the argument of the evolution of the species, and must have equally influenced our brain circuitry. Thus, both the human and the animal brain must be capable of dialectic activity that is useful for the individual's rapport with nature, as is its analytical capacity. Therefore, dialectic logic must be hidden among historical thought, linguistic structures and science.

The search for dialectics then becomes the search for *non-logic*, failures and cracks in the apparently monolithic proposal of traditional logic.

## The dialectics of *yin–yang*

In traditional Chinese philosophy (but without making reference to any specific author) we can frequently find the classification of all the elements in the universe–from foodstuff to human attitudes–divided into two categories. The words *yin* and *yang* designate the final objective of the structural homomorphism of traditional Chinese thought:

> [ ... ] *les notions secondes de yin et de yang deviennent des*

> *entités scolastiques que la spéculation utilise pour ordonner les faits. Le yin et le yang cessent d'être des principes concrets ; pourtant l'orientation dualiste qu'ils ont donnée à la pensée est un fait acquis. Ni le yin ni le yang ne deviendront par eux-mêmes des réalités religieuses, mais un parti pris de classification bipartite continuera à dominer le monde des choses sacrées : l'âme restera double* [ ... ][55] [30]

The dialectics of Lao-Tze are very simple and are based on the binary organization of the opposites, *yin* and *yang*, of the Chinese philosophical tradition.[56] From the beginning, we read:

> [ ... ] So it is that existence and non-existence give birth the one to the other; that difficulty and ease produce the one the other; that length and shortness fashion out the one the figure of the other; that height and lowness arise from the contrast of the one with the other; that the musical notes and tones become harmonious through the relation of one with another [ ... ] [53, II]

Not much speculation is needed in order to see here the "unity and struggle of opposites" which would be used by Hegel, Marx and Engels centuries later.

A reader who is familiar with Chinese logic and thought will perhaps hesitate to accept that these ideas can be considered *logical values* and not "metaphysical entities" regarding which there is nothing to explain. This doubt will appear more than once in the upcoming sections, but we will delay its analysis until we have all the necessary elements.

There is no one better to illustrate the two logical values than Arnold J. Toynbee (1889, 1975), since the notions of *yin* and *yang* are the main pillars in his analysis of historical movement. In the following

[55] [ ... ] the secondary notions of yin and yang turn into scholastic entities that speculation uses to bring order to facts. Yin and yang cease to be concrete principles, but the dualistic orientation they have given to thought is a consummated fact. Neither yin nor yang will become religious entities, but the bipartite classification will continue to rule the world of the sacred: the soul will remain double [ ... ].

[56] In chapter 42, Lao-Tze makes indirect mention of the opposite principles yin and yang. However, Zhuangzi repeatedly and explicitly mentions them.

selected fragments of his entire works–a compendium of Somervell's abridgment–we find:

> This alternating rhythm of static and dynamic, of movement and pause and movement [ ... ] the sages of the Sinic Society described these alternations in terms of Yin and Yang [ ... ] In every case the story opens with a perfect state of Yin. [ ... ] When Yin is thus complete, it is ready to pass over into Yang. [ ... ] History duly reveals to us in the phenomena of disintegration a movement than runs through war to peace; through Yang to Yin; [ ... ] The work of the Spirit of the Earth [ ... ] manifest itself in the geneses and growths and breakdowns and disintegration of human societies [ ... ] we can hear the beat of an elemental rhythm whose variations we have learn to know as challenge-and-response, withdrawal-and-return, rout-and-rally, apparentation-and-affiliation, schism-and-palingenesia. This elemental rhythm is the alternating beat or Yin and Yang [ ... ] the movement that this rhythm beats out is neither the fluctuation of an indecisive battle nor the cycle of a treadmill. The perpetual turning of a wheel is not a vain repetition if, at each revolution, it is carrying the the vehicle that much nearer to its goal; [ ... ] On this showing the music that the rhythm of Yin and Yang beats out is the song of creation [ ... ] If we listen well we shall perceive that, when the two notes collide, they produce not a discord but a harmony. Creation would not be creative if it not swallow up all things in itself, including its own opposite. [93][57]

For Toynbee, *yin* and *yang* are not only a metaphor but represent organized thought. Throughout the volumes of *A Study of History* [92] he will resort to these notions over and over again–as a Chinese scholar would–in order to interpret the movement of history. In the fragments

[57] This quote has been assembled using fragments from [II, iv, 2], [II, v, 1], [V, xvii, 2] and [V, xxii].

cited here we can find the unmistakable mark of *yin-yang* dialectics explaining movement and containing the idea of opposites, of turning around oneself, of progressing with each turn.

We will see Toynbee's logic in more detail. Toynbee states–and the context always confirms–that each historical proposition is either *yin* or *yang*. This is due to the fact that each historical moment can be classified as either *dynamic* or *static*. For Toynbee, there are states in society and therefore, these can be transferred to statements on history. This is a delicate point in our study. Let us consider a historical statement. According to Toynbee, this statement will be valid within a certain context and in either a *yin* or *yang* period, depending on the case. Historical truths do not appear as universal truths, but as valid truths only within one historical period. The *yin* statement is succeeded by the *yang* statement and vice versa. The behavior of reality forces the behavior of the logic used in its study.

The transformation of yin and yang between themselves, the so-called *alternating beat* or *song of creation* is none other than a logical function that materialists have dubbed *negation*. Negation appears as the mechanism of movement and as a basic process in the course of history. This is also a close point of contact with materialist dialectics' notion of becoming. The idea is expressed–by using the symbol $\rightarrow$ to indicate becoming–as:

$$\cdots \rightarrow yin \rightarrow yang \rightarrow yin \rightarrow yang \rightarrow \cdots$$

The most simple, direct and most stimulating way to interpret the yin-yang dialectics was introduced by Oscar Wilde (1854, 1900) in an essay on art:

> For in art there is no such thing as a universal truth. A truth in art is that whose contradictory is also true. [96]

It is difficult to find so many logical ideas and with such content, in so concise a manner. Of course, this statement offers only a glimpse of Wilde's wit, but we will attempt to see much more. Let us assume that the statement can be taken in its strict literal sense. Then, there are at least three types of logical values at play:

- universal truths, openly mentioned;
- (universal) falsehoods, by opposition;
- truths in Art, openly mentioned.[58]

This passage conveys that there is an associated concept of negation binding the groups of logical values. Figure 1 offers a double diagram which interprets these facts.

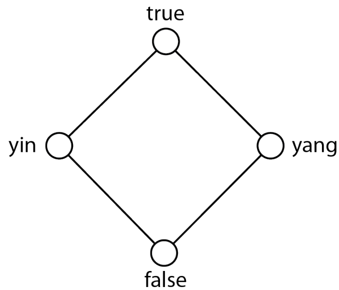


Figure 1: *Yin-yang* dialectics lattice.

In the diagram (a lattice in its mathematical sense), Wilde's new logical values are referred to as *yin* and *yang*. These names–which we will use temporarily and later abandon–are taken from traditional Chinese philosophy and their choice is justified in what follows.[59] The notion of *negation* introduced by Wilde, and the operation of *negation* according to Toynbee–the "song of creation"–, respectively assigning 1 and 0 to *true* and *false*, can be expressed as:

$$N = (0\ 1)(yin\ yang)$$

[58] In an epigram, Wilde suggests the antagonism between realistic and romantic literature: *The nineteenth century dislike of realism is the rage of Caliban seeing his own face in a glass. The nineteenth century dislike of romanticism is the rage of Caliban not seeing his own face in a glass.* There is nothing clearer than the statement of this contradiction. Today, we might add many other literary contradictory genres to these two: socialist realism, magical realism, historical literature, ucrony, Ergodic literature and others that I leave out or have not appeared yet.

[59] This is a well-known lattice which matches the $2^{nd}$ order Boolean lattice designated as $\mathbf{B}^2$ –in the notation proposed in this study– or lattice **D2**, which will be defined further ahead.

where here I use–as in the rest of the book–the usual notation of substitution in group theory.[60] This means that the negation $N$ transforms 0 ito 1 and reciprocally, *yin* into *yang*, and reciprocally.

Wilde's original statement acquires great logical precision in this context and cannot be considered just a witty phrase. Aside from the universal truths and falsehoods which apply to, for instance, mathematics, science or technique, statements on art suffer a different fate. With the exception of statements which can be trivially true or false, all the other statements have a different logical value. Wilde's cited statement can be precisely reformulated as:

> For in art, every non-trivial statement possesses *yin* or *yang* logical value. The negation of every truth in art is an equally valid statement.

Based on the previous, we understand that *yin* and *yang* logical values are values which indicate truthfulness and not falsehood, although this truth can be *partial* or *limited*. Since this is the first example of a statement which fails to use traditional logical values, it is appropriate to delve on this a bit more. As an example, let us consider the two statements:

- art is a reflection of reality,
- art creates a new reality.

Most people will agree with the following statements:

- These statements are–in some way–opposed.
- These statements are not–in any way–universally true or universally false; in any case, establishing their logical value is not a simple thing to do.
- The first statement has a materialistic tone, while the second has an idealistic tone.

Conversely, getting consensus on the following statements is unlikely:

[60] This notation is described further ahead, see page 106.

- The first statement is *yin* and the second is *yang*.
- The first statement is markedly "masculine" while the second is markedly "feminine'.
- The first statement is markedly "static" while the second is markedly "dynamic".
- None of the two statements has some truth, they must both be considered equally false.
- The opposite of the preceding statements.

The response to these possibilities will be analyzed after obtaining further input from *yin-yang* dialectics. These considerations show that the human brain is able to work with logical values other than "true" or "false" and that these concepts are applicable to the universe and have real interest. We will use plenty of examples of *yin-yang* dialectics with the purpose of establishing this idea more firmly.

Wilde's clever remarks, Chinese scholastic thought, or Toynbee's study of history are not the only historical examples of *yin-yang* dialectics. Freud's sexual theory is yet another extremely interesting example of this logical structure:

> Sadism and masochism occupy a special position among the perversions, since the contrast between activity and passivity which lies behind them is among the universal characteristics of sexual life [ ... ] its active and passive forms are habitually found to occur together in the same individual [ ... ] We find, then, that certain among the impulses to perversion occur regularly as pairs of opposites; and this, taken in conjunction with material which will be brought forward later, has a high theoretical significance [ ... ] Whenever we find in the unconscious an instinct of this sort which is capable of being paired off with an opposite one, this second instinct will regularly be found in operation as well. Every active perversion is thus accompanied by its passive counterpart [ ... ] [26]

*Pairs of opposites*, per Freud's words, quickly lead to logical values for making statements on human behavior. It is possible to distinguish four different logical values:[61]

- universally valid statements,
- valid statements for the unconscious,,
- valid statements for the conscious mind,
- false statements.

These four values, combined with the condition of pairs of opposites, lead to *yin-yang* dialectics. Following this line of thought, *negation* plays a crucial role. The process by which a statement of the conscious mind is transferred to the unconscious is a *negation*. The operation which performs the opposite change is also a *negation*. The first process is linked to the "genesis of neurosis"; the second, with "therapy". As is well known, the work of the therapist–according to Freud–consists in turning the truths of the *unconscious* into truths of the *conscious* mind. In our logical language, the operation at hand is also a *negation*. It is even eloquent to state that "the negation of the conscious attitudes is the basis of neurosis", while "therapy consists in the negation of the unconscious content".[62]

In the cases we have presented, *yin-yang* dialectics appear spontaneously. There are no attempts by the cited authors, not even a hint of suspicion that we are in presence of a new mechanism of reasoning. In all cases, this new way of thinking presents itself as dialectics and not as Boolean logic, despite the fact that they are formally equal.[63] How they handle contradiction and negation shows as much. But above all, there is an even more powerful reason to know that we are not in presence of Boolean logic as derived from logical binary combinations. The

[61] Strictly speaking, Freud's subsequent analysis led him to consider that the unconscious presented two struggling entities: the *Super-ego* and the *Id*. There is an outline of Hegel's dialectics at play here, even if it was not considered so by the author.

[62] It is to be expected that this manner of presenting old results does not inaugurate a new school of psychoanalysis.

[63] This statement is very delicate and is clarified with what follows. In reality, the *yin-yang* dialectics is homomorphous with binary logic.

conversion between $\mathbf{B}^2$ and $\mathbf{B}$ can be done through the mechanism of introducing or eliminating fictitious variables. In none of these cases is this procedure suggested. We cannot say that there is a hidden binary variable allowing us to separate contradictory truths in art, history or the unconscious. The introduction of fictitious variables would be a way to "save face", in the scholastic manner, or of creating a "conventional" logic, as Henri Poincaré would have put it.

In hopes of complementing the above, two quotes by the classics of dialectic materialism are of interest, as they are directly linked to *yin-yang* dialectics. We will begin with the–imprecise[64]– statement formulated by Engels on one of the laws of dialectics:

> *Alle Naturvorgänge sind doppelseitig* [ ... ] [21, Artikel, Grundformen der Bewegung][65]

If we believe this statement is general in nature and that **D2**, this statement translates the results we have found and speaks of "two-sided" symmetry.

The second quote belongs to Lenin–Vladimir Ilich Ulianov–(1870, 1924), and touches on an interesting point. Due to the nature of the quote itself, it may occur that readers who are familiar with historical materialism may feel a bit disoriented, but throughout the course of this study, the interpretation will increase in precision. Lenin says:

> [ ... ] one must not fail to see [ ... ] the struggle of parties in philosophy, a struggle which in the last analysis reflects the tendencies and ideology of the antagonistic classes in modern society [ ... ] [as the struggle between] materialism and idealism. [55]

This way to study philosophy resembles *yin-yang* dialectics. Here, we can say that any philosophical thesis is not a universal truth but has one of two logical values: *materialism* or *idealism*. To use Wilde's terminology, the opposite of a valid philosophical statement is also a valid

[64] We will come back to this quote further ahead.

[65] All of the processes of nature are two-sided. [21, Articles, Foundations of movement]

philosophical statement. One is idealist in nature and the opposite is materialistic in nature. Within the realm of dialectic thought, it is possible to understand this duality and contemplate it from a more general standpoint.

We can now return to these *yin* and *yang* logical values. According to Chinese thought, Toynbee's interpretation of history has the following connotations:

*yin* = the static, the passive, the femenine,

*yang* = the dynamic, the active, the masculine.

However, there is good reason to not accept these identifications at face value. Firstly, since there is total symmetry in the **D2** lattice, this collides with the possibility of actually distinguishing *yin* from *yang* values. Secondly, it is rather meaningless to try to qualify logical values if not for their formal characteristics. Thirdly, it is not long before we encounter trouble. Let us consider Freud's case as an example: not only can we find the *static* and the *dynamic*, but the *conscious* and the *unconscious*, and it is not possible to identify ourselves with one or the other value. For these reasons, we cannot accept that there is an *internal and unchanging* meaning of the values in this dialectic. Conversely, we accept that these values, *from a formal point of view*, are indistinguishable.

If we accept that there is no meaning in trying to endow the *yin* and *yang* values with absolute meaning, we can go through the main ideas behind this logic. Two negations are present in this lattice:

$$N_1 = (0\ 1) \qquad N_2 = (0\ 1)(yin\ yang)$$

This indicates that by means of a negation 0 it is possible to operate a transformation into 1 and reciprocally, and it also indicates that *yin* becomes *yang*, and the latter becomes *yin* once again. Each list between brackets indicates a closed cycle. If some element is missing, it means that it is transformed in itself by the operation.

At this point in the analysis, we use the notion of negation spontaneously, with no greater critical analysis. We will analyze this important subject in more depth further ahead.

The first negation, $N_1$, does not affect the *yin* and *yang* values, it only exchanges the endmost values of the lattice. The second negation, $N_2$, matches the usual negation in $\mathbf{B}^2$. The main difference between the $\mathbf{B}^2$ Boolean logic in its traditional interpretation and **D2** *is the existence of two negations instead of a single one*. The two diagrams are identical.

From the point of view of its applications, statements are divided into two main types: *universal* statements (true or false) and *dialectic* statements. Mathematical or logical statements, which are pure formal values, belong to the first type in all the contexts analyzed. Statements on art, history, psychological behavior or philosophy belong to the second type, as per the examples analyzed. In the final chapter we will see that dialectic values also exist in the natural sciences. Negation operations have a different meaning depending on the field of knowledge being considered, but a negation is always associated with a mechanism of change or action.

The reason for going through this rudimentary form of dialectics further lies in that it allows us to clarify many of the ideas presented in this study. But far from exhausting the subject, this section has only just begun to analyze the problem of the foundations of dialectics.

## The dialectics of Vico and Hegel

The notion of dialectics has also been introduced in the social sciences. We will begin by studying one of the heralds of this topic.

Giambattista Vico (1668, 1744) is the author of a lengthy essay on human history from 1725, *La scienza nuova* [94], where he presents an argument regarding the existence of historical laws and the cyclical interpretation of history: *teoria dei corsi e dei ricorsi storici* (theory of historical advances and reversals). Human history comprises the cyclical repetition of three states: *l'età degli dei* (the age of the gods), *l'età degli eroi* (the age of heroes), *l'età degli uomini* (the age of men). In the first age, society is theological in nature. In the second, aristocratic; in the third, it is a society of equals. There is no doubt that, for Vico, three historical principles–a "new science", such as he named his work–are clearly established:

- history has laws, just like science;

- the becoming of history is cyclical in nature;
- it comprises three stages.

In this, we cannot miss the beginnings of dialectic materialism which will be later developed by Hegel.[66]

Hegel's dialectics are the first example of non-binary logic to be stated as such. Given that it is the most important case of dialectics, this section will only introduce the topic. The entire work revolves around this dialectic and its generalizations, so that we will return over and over to the issue of its interpretation.

Hegel was the first logician to propose the need for three additional logical values, aside from*true* y *false*. These three logical values were originally presented by Hegel as instances of knowledge. Subsequently, the German dialectic materialists–Marx y Engels, ver [21, 22, 62]– expanded the scope to include laws that described instances in the movement of history. These are, as is well known:[67]

*thesis* = the starting point

*antithesis* = negation of the previous

[66] There are other proposals on this matter. In [14] Auguste Comte (1798, 1857) established three stages to the development of philosophy: *d'abord la méthode théologique, ensuite la méthode métaphysique, et enfin la méthode positive* (first the theological method, then the metaphysical method and finally the positive method). In [66] Lewis H. Morgan (1876, 1950) defined 10 stages in the evolution of human society.

[67] In a strict sense, traditional Chinese philosophy had already warned of the existence of three values. This is expressly manifest in how *yin-yang* is used with regards to food. All foodstuffs are classified as either *yin* or *yang*. However, it was clear that it was necessary to add a third category for *neutral* elements, those which were neither one nor the other. The clearest example can be given with tea types: green tea is simply made up of dry leaves, black tea is made from fermented leaves, and a third kind of tea, blue or *oolong* tea, is partially fermented and partially dry. This idea extended to all types of foodstuffs. It is interesting to note that some classical Chinese scholars observed an absence within the *yin-yang* duo, which is why it is not infrequent to find the *yin-yang-dao* triad. This is undoubtedly one of the clearest previews of Hegelian logic. Traditional Indian philosophy also discovered an incipient form of these ternary dialectics. The Brahma, Shiva and Vishnu triad expresses a triple aspect of the dynamics of the universe: *creation*, *destruction* and *conservation*. This triad structure did not reach the level of abstraction of Chinese or Western thought, but it exuberantly expresses this triple notion.

*synthesis* = negation of the previous and point of arrival

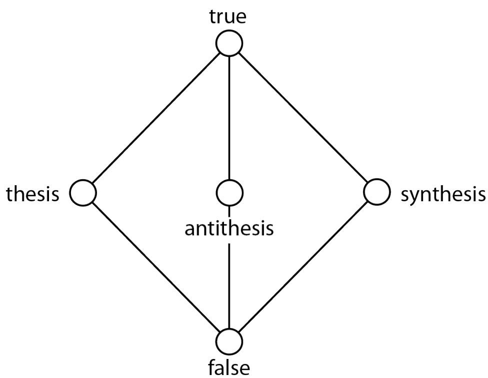


Figure 2: **D3** lattice of Hegelian logic.

Figure 2 presents the diagram corresponding to Hegel's dialectics.

## Materialistic dialectics

Friedrich Engels was often preoccupied with the materialistic interpretation of Hegel's laws. The reality of the universe demands that, aside from studying matter itself, its movement be studied from a scientific standpoint. If we were to assume, as we have until today, that binary logic (or Aristotelian or classical logic, as we will many times refer to it) is a reflection of the general laws of matter, dialectic logic then corresponds to the general laws of the movement of matter.

> *Es ist also die Geschichte der Natur wie der menschlichen Gesellschaft, aus der die Gesetze der Dialektik abstrahiert werden. Sie sind eben nichts andres als die allgemeinsten Gesetze dieser beiden Phasen der geschichtlichen Entwicklung sowie des Denkens selbst. Und zwar reduzieren sie sich der Hauptsache nach auf drei: das Gesetz des Umschlagens von Quantität in Qualität und umgekehrt; das Gesetz von der Durchdringung der Gegensätze; das Gesetz von der Negation der Negation. Alle drei sind von Hegel in seiner idealistischen Weise als bloße* Denkgesetze *entwickelt* [ ... ] *Der*

> *Fehler liegt darin, daß diese Gesetze als Denkgesetze der Natur und Geschichte aufoktroyiert, nicht aus ihnen abgeleitet werden.* [21, Artikel, Dialektik][68]

Sadly, Engels' text only directly analyzes the first law of transformation of quantity into quality. This law establishes that the accumulation of quantity is at the root of change. It is not a formal, but a material law, and for this reason, this work will only indirectly make reference to it. The following section analyzes this point.

The second law, the law of penetration of opposites, establishes the following:

> *Alle Naturvorgänge sind doppelseitig, beruhen auf dem Verhältnis von mindestens zwei wirkenden Teilen, auf Aktion und Reaktion.* [21, Artikel, Grundformen der Bewegung][69]

That is all. When we analyze reality, it leads to two aspects which are presented as different, opposed, contrary: the two poles between which movement unfolds. The search for these opposites is not a simple task and cannot be taken lightly.

The third law of dialectics–undoubtedly the richest, formally speaking–establishes that the interplay of opposites continuously takes us back to situations we have already experienced, but in an enriched, enhanced manner. Movement has three consecutive phases: the starting point, the negation of the starting point and the return to the starting point; negation of the negation. The third law of dialectics regulates the cause of movement. Its "becoming" statement is as follows:

$$\text{thesis} \rightarrow \text{antithesis} \rightarrow \text{synthesis}$$

[68] The laws of dialectics derive, therefore, from the history of nature and the history of human society. These laws are not, in fact, anything else than the most general laws of these two phases of historical development and of thought itself. And they are reduced, fundamentally, to three: the law of transformation of quantity into quality and vice versa, the law of penetration of opposites, and the law of negation of the negation. The three laws have been developed by Hegel–in his idealistic manner–as simple laws of thought [ . . . ] The error lies in that, as laws of thought, they are imposed upon nature and history, instead of being drawn from these. [21, Articles, Dialectics]

[69] All the processes of nature are two-sided, since they rely on the relationship between at least two interacting parties, action and reaction. [21, Articles, Foundations of movement]

This statement does not appear directly in Engels' exposition, which is an unfinished draft [21], but it is used, for instance, in *Das Kapital* [60] and also by Hegel [42].

If we consider the diagram in Figure 2, we can analyze the Hegelian negation. It is clear that, by definition, we can introduce this same statement through the expression:

$$N = (0\ 1)(\text{thesis antithesis synthesis})$$

Which results in the following expressions:

$$N\,\text{thesis} = \text{antithesis} \qquad N\,\text{antithesis} = \text{synthesis}$$

And, maybe not so clearly, in the conditions:

$$N\,\text{synthesis} = \text{thesis} \qquad N\,N\,\text{thesis} = \text{synthesis}$$

In the non-formal expositions of dialectics it is not clear that the negation of the *synthesis* is a new *thesis*. Somehow, it is usual to believe that the logical value *synthesis* matches that of *thesis*. If this were the case, dialectics would become *yin-yang* dialectics, which is utterly false. For the first time we find a result which, when precisely stated, acquires an unexpected feature. This situation will appear many other times.

It is important to analyze the problem of negation. In Hegelian logic, negation is grade 6, while in *yin-yang* logic it is grade 2. For instance, this difference makes the double negation of the *thesis* non-trivial, unlike what happens in the **D2** *yin-yang* logic:

$$N\ N\ \text{yin} = \text{yin} \qquad N\ N\ \text{yang} = \text{yang}$$

Conversely, the triple negation in **D3** is verified with: $N\ N\ N\ \text{thesis} = \text{thesis}$.

As in the case of *yin-yang*, logical values are questioned in the Hegelian case. To a large extent, it is difficult to accept, at least at first, that the classical terminology would make reference to logical values and not a different type of entity. In the imprecise formulations of dialectics is it usually considered that *thesis*, *antithesis*, etc., are states in a dynamic process. Something equal to the case of *yin-yang* dialectics occurs: it is the material properties which generate logic, and hence the confusion. In all, the problem stems from the fact that logic is a *reflection*, *an abstraction* of the material properties of the universe. This is the reason for the confusion.

There is yet another issue worthy of note which may cause some initial surprise. As in the *yin-yang* diagram, the three elements cannot be distinguished from one another in Hegel's. This makes a *thesis* practically undistinguishable from an *antithesis* or a *synthesis*. In an abstract sense, a statement does not have one of these values *assigned by its content*. The three values are equally applicable. This means that any statement can either be a starting point or a point of arrival, both a *thesis* and an *antithesis*–the differences stem only from their reciprocal relationships..

Ultimately, Toynbee's theoretical idea of history differs from that of historical materialism in only one aspect: whereas the first occurs within the dialectic logic of two elements, the second takes place within the dialectics of three elements. This is, of course, quite a reduced way of stating abysmal differences, but it is good to point out that, in abstract, the difference is numerical.

We will come back to this further ahead. In the meantime, we will introduce some more elements related to the interpretation of the logical values of dialectics.

## The first law of dialectics

Engels' text on the first law of dialectics is clear and offers no difficulties in its understanding:

> *Gesetz vom Umschlagen von Quantität in Qualität und umgekehrt. Dies können wir für unsern Zweck dahin ausdrücken, daß in der Natur, in einer für jeden Einzelfall genau feststehenden Weise, qualitative Änderungen nur stattfinden können durch quantitativen Zusatz oder quantitative Entziehung von Materie oder Bewegung* [ . . . ] [21, Artikel, Dialektik][70]

[70] Law of transformation of quantity into quality and inversely. For our purposes we can express this law by saying that, in nature, and in a clearly established manner for

This law regulates the "leaps" in the process of becoming. It very clearly establishes that every change always obeys to a quantitative accumulation of an actual, measurable and observable entity. This accumulation leads to a "leap", a change in quality. At the same time, after the change, a–possibly different–process of accumulation begins once again, and so on.

In everyday language this idea is expressed in several ways. For example, we say:

- the straw that broke the camel's back
- the pitcher goes so often to the well that it is broken at last

In science there are several examples of the application of this law. One of the first examples was formulated by the famous alchemist Paracelsus with regards to poisons, and by extension, remedies. It constitutes one of the bases of pharmacology:

> *Alle Dinge sind Gift, und nichts ohne Gift; allein die Dosis macht, daß ein Dinge kein Gift ist.*[71]

Without question, remedies and poisons are two opposing ideas. However, according to this statement, it is only the amount which actually prompts the change.

Henri Poincaré (1854, 1912) was the first to discover an example that constituted a precise leap in quality for mathematics. The movement of the two particles gravitating between each other had already been resolved by Newton. However, it was Poincaré who discovered that, for three or more particles, the problem was of great mathematical complexity. This example gave way to new problems and new study methods for dynamic systems, see [81].

Physics has several examples of the application of this law, but let us consider one of the most straightforward: statistical mechanics. As

---

each specific case, qualitative changes can only be produced through the quantitative addition or subtraction of matter or movement [ . . . ] [21, Articles, Dialectics]

[71] All things are poison and nothing is without poison; only the dose makes something not poison.

an example, we will take a quote from theoretical physicist Lev Landau (1908, 1968) and Evgeny Lifchitz (1915, 1985) in the early days of statistical mechanics:

> *Ainsi, quoique le mouvement d'un système mécanique ayant un grand nombre de degrés de liberté soit soumis aux mêmes lois mécaniques que le mouvement d'un système ayant un petit nombre de particules, la présence même de ce gran nombre de degrés de liberté donne naissance à des lois qualitativement nouvelles.*[72] [52, I, 1]

The behavior of a system formed by a number of equal elements depends on the number of elements. Let us assume that we have $N$ number of elements. If $N$ is small, the system will simply be the sum of its parts and its global behavior will be able to be analyzed and explained through the behavior of each of its parts. But if we were to increase the number $N$, we would reach a point where the system could no longer be explained through its constituent parts. It will have acquired new properties by the mere fact of exceeding a boundary of quantity. The study of these systems gave way to a new branch of physics: statistical mechanics.

Darwin's evolution of the species is based on the accumulation of small favorable differences which end up creating a new species. This happens, for example, due to a change in the environment or due to migration and adaptation to a new environment.

In the social sciences, quantity also determines quality. We will consider the number $N$ of human beings which are the object of study. For example (using arbitrary limits), if $N < 6$, it is studied by psychology, if $6 \leq n \leq 50000$ it is the object of study of sociology or anthropology, and if $N$ is much larger than these numbers, it is the object of history. Of course, the proposed limits are conventional and only exemplify a problem similar to statistical mechanics.

In economics, John Maynard Keynes (1883, 1946) argued that in times of crisis, the behavior of individuals is one of saving, but that

[72] Although a system with a large number of degrees of freedom is subject to the same laws of movement as a system with a small number of particles, the presence of this great number gives way to qualitatively new laws.

the behavior of the State should be one of investment. In this case, it is clear that there is a change in quality by an increase in the affected social group. This is sometimes jokingly referred to as "saving money builds poverty", as opposed to the usual statement which conveys the opposite.

Problems leading to this law of dialectics also appear in philosophy. Perhaps the best example is one attributed to Bertrand Russell which intends to shed light on the weakness of mere experimental observation and empirical laws. The example is as follows:[73]

> It concerns a turkey who noted on his first morning at the turkey farm that he was fed a 9 am. After this experience had been repeated daily for several weeks the turkey felt safe in drawing the conclusion "I am always fed at 9 am". Alas, this conclusion was shown to be false in no uncertain manner when, on Christmas eve, instead of been fed, the turkey's throat was cut. [13, IV]

The example is witty, but being an empiricist is not the same as being a dialectician. The turkey's analysis is incomplete. A more detailed observation would have shown him that, aside from being fed every day, he *systematically gained weight*. A dialectic turkey would have thought that "something is going on, I'm gaining weight, this can't go on indefinitely like this, something is bound to happen, I don't know what or when, but something new will happen". This example and its two interpretations show the difference between a simple and a dialectic analysis.

## Ionian dialectics

In the $6^{th}$ century BC, in Ionia, a major concept for Western thought was born: the idea of the elements. In its traditional form, matter was formed by four elements: *fire*, *earth*, *air* and *water*. Among the fragments preserved from Empedocles we find:

[73] This idea is outlined in [87, VI].

> Hear first the four roots of all things: shining Zeus, life–bringing Hera, Aidoneus and Nestis [ . . . ] [20, Diels #6]
>
> At one time it grew together to be one only out of many, at another it parted asunder so as to be many instead of one; Fire and Water and Earth and the mighty height of Air; [ . . . ] [20, Diels #17]
>
> For all of these –sun, earth, sky, and sea– are at one with all their parts that are cast far and wide from them in mortal things. [20, Diels #22]
>
> For they prevail in turn as the circle comes round, and pass into one another, and grow great in their appointed turn. [20, Diels #26]

It is usual to interpret these in a literal sense and make the Ionian materialists appear to have said that the entire universe is made up of these four entities. However, a more interesting interpretation is possible. The last fragment by Empedocles opens a door, and Heraclitus offers his interesting perspective:

> Fire lives in the death of earth, air in the death of fire, water in the death of air, and earth in the death of water. [45, Diels #34]

The idea of *living in the death of* (something), of an evidently dialectic nature, can be found in other fragments by Heraclitus which have survived. This statement has a circular becoming structure:

$$\cdots \to \text{earth} \to \text{fire} \to \text{air} \to \text{water} \to \text{earth} \to \cdots$$

The idea of becoming is present in several fragments by Heraclitus:

> They do not step into the same rivers. It is other and still other waters that are flowing. [45, Diels #20]
>
> You cannot step twice into the same river, for other waters and yet others go ever flowing on. They go forward and back again. [45, Diels #21]

> Into the same rivers we step and do not step. We exist and we do not exist. [45, Diels #110]
>
> Cool things become warm, the warm grows cool, the moist dries, the parched becomes moist. [45, Diels #22]
>
> Immortals become mortals, mortals become immortals; they live in each other's death and die in each other's life. [45, Diels #66]
>
> In the circumference of the circle the beginning and the end are common. [45, Diels #109]

If we interpret the idea of rotation of the elements as a negation, these may be considered in diagram form, as shown in Figure 3.

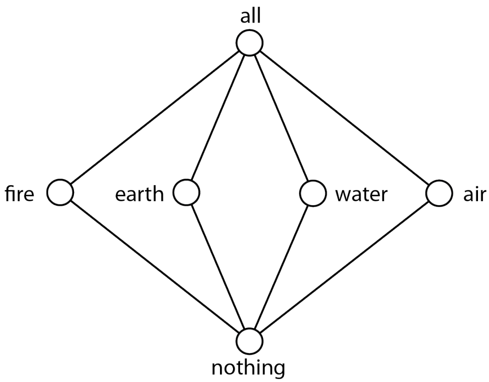

Figure 3: **D4** lattice of the elements in Ionia.

The endmost values in the diagram can be interpreted in terms of the elements, which correspond to the void and the *quintessence* or *ether*. A negation is defined on this lattice. Due to its similarity to the dialectic cases we have already found, the negation is:

$$N = (nothing\ all)(earth\ fire\ air\ water)$$

once again employing the notation of substitutions. We then have $N\ earth = fire$, $N\ fire = air$ and so on.

So far we might think there is more fantasy to this than reality. However, subsequent medieval scholastics added other four basic notions to the four traditional elements: *moist*, *cold*, *dry* and *hot*. It is well-known that logical relations such as the following have been established:

*water* = *moist* **AND** *cold*

*earth* = *cold* **AND** *dry*

*fire* = *dry* **AND** *hot*

*air* = *hot* **AND** *moist*

These relations lead to the diagram we will refer to as **2D4**, which is presented in Figure 4. It is interesting to note that the new logical relations match the rotation of the elements proposed by Heraclitus.

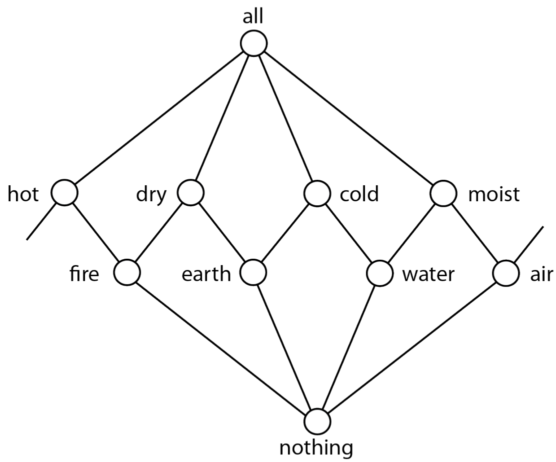

Figure 4: **2D4** lattice of the medieval elements.

## The elements in China

Traditional Chinese thought understands the elements in a manner different from that of the West:

> *Les Éléments étant énumérés dans l'ordre de la succession des Saisons qu'ils symbolisent, la théorie veut que cet ordre soit celui d'une succession régulière en forme de cycle. D'après cette théorie, dite de la production réciproque des éléments, le*

*Bois engendre le Feu, le Feu engendre la Terre* [ ... ] *l'Eau engendre le Bois. Une troisième disposition oppose les Éléments* [ ... ] *La théorie correspondante est celle d'après laquelle les éléments triomphent les uns des autres dans l'ordre inverse à celui de l'énumération : le Métal triomphe du Bois , le Bois de l'Eau* [ ... ] *la Terre du Métal.* [30][74]

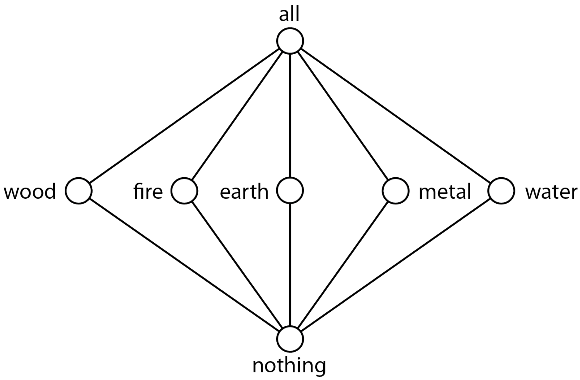

Figure 5: **D5** lattice containing the Chinese elements.

The five Chinese elements, which clearly differ from the Western elements, are related by two manners of turning, *two negations*, as is the appropriate way to put it in a dialectic conception. The first negation is associated with genesis, the second to destruction. As we can see, generally speaking, these ideas extend Heraclitus' thought. We can interpret this theory as a **D5**, Figure 5, in which two different negations are considered, genesis **G** and destruction **D**:

$$G = (nothing\ all)(wood\ fire\ earth\ metal\ water)$$

$$D = (nothing\ all)(metal\ wood\ water\ fire\ earth)$$

[74] The elements are numbered in the order of succession of the Seasons they symbolize. The theory attempts to support that this order is that of a regular succession in the manner of a cycle. According to this theory, the theory of the reciprocal production of the elements, Wood engenders Fire, Fire engenders Earth [ ... ] Water engenders Wood. A third disposition opposes this [ ... ] The corresponding theory is that in which the elements win, the ones over the others [ ... ] Metal wins over Wood, Wood wins over Water [ ... ] Earth wins over Metal.

These two transformations give way to different "rotations" in the lattice and also admit a *becoming interpretation*, as in the previous cases. This is a refined discovery we will further ahead have an opportunity to come back to.

Daoist thought tried to reconcile the binary idea of *yin* and *yang* with the five traditional elements. Zhuangzi–one of the most important Daoist masters, see [97, 98, 99, 100]–links Daoist thought to the traditional idea of the *yin-yang* duality, something suggested by Lao Tze yet not explicitly mentioned:

> When the state of Yin was perfect, all was cold and severe; when the state of Yang was perfect, all was turbulent and agitated. The coldness and severity came forth from Heaven; the turbulence and agitation issued from Earth. The two states communicating together, a harmony ensued and things were produced. Someone regulated and controlled this, but no one has seen his form. Decay and growth; fullness and emptiness; darkness and light; the changes of the sun and the transformations of the moon:– these are brought about from day to day; but no one sees the process of production. Life has its origin from which it springs, and death has its place from which it returns. Beginning and ending go on in mutual contrariety without any determinable commencement, and no one knows how either comes to an end. [97, XXI, 4]

We see here a complex statement in which *yin* and *yang* interpenetrate and undergo a circular motion which generates everything and causes everything. Daoist thought, then, had to reconcile the elements with this causality. With five elements, it is enough to accept that the process of synthesis has only partial stages. The elements' genesis cycle turns now, with the dialectic vision, into:

$$\begin{array}{ccccccccc} \text{metal} & \rightarrow & \text{water} & \rightarrow & \text{wood} & \rightarrow & \text{fire} & \rightarrow & \text{earth} \\ \text{new } \textit{yin} & \rightarrow & \text{full } \textit{yin} & \rightarrow & \text{new } \textit{yang} & \rightarrow & \text{full } \textit{yang} & \rightarrow & \text{synthesis} \end{array}$$

This expanded form of dialectics is one of the pinnacles of Chinese thought, aside from preceding Hegel in over two thousand years.

The major conclusion and question that this section poses–as well as the two previous–, has shifted from interpreting the diagrams to a more general problem regarding the meaning of logic and its connection with the material reality of the universe.

## Dialectics in pre-Columbian America

In pre-Columbian America we find traces of an incipient dialectic thought. We do not have much information because only the Mayans had devised an elaborate writing system. Direct oral transmission was also rare since conquerors, chroniclers and settlers were not interested in becoming acquainted with the natives' way of thinking. In spite of this, it can be detected at least twice: in the *Anasazi* in North America and the *Aymara* in South America.

The idea of the universe among the *Anasazi* is based on the existence of a cosmic correspondence, see [40], in which there is a link between the spatial directions, colors, totemic animals and other natural phenomena: the trees, the seasons and the "elements". Table 2, see [7], shows the specific case of the *Zuni*.[75] Undoubtedly, the existence of this cosmic correspondence–which can also be found between the Chinese and the Mongolian, for example–evidences a high degree of abstract philosophical thought.

Table 2: Cosmic correspondence among the *Zuni*.

| **direction** | **color** | **totem** | **season** | **"element"** |
|---|---|---|---|---|
| North | yellow | lion | winter | wind |
| West | blue | bear | spring | water |
| South | red | badger | summer | fire |
| East | white | wolf | fall | frost |
| Zenith | all colors | eagle | | |
| Nadir | black | mole | | |

[75] Among the *Tewa*, the correspondences are somewhat different. Among the *Hopi* there are also correspondences and it is possible that all the *Anasazi* had them as well. The *Aztecs* had a similar correspondence, but only in relation to the four directions, which have many elements in common with Table 2. This is only natural since there is a close connection between the different peoples.

The idea of negation or becoming is not described. Something worthy of note is that *six* "elements" are involved, which is somewhat more complex than among the Chinese or the Greek..

The most surprising case of natural logic is, without question, the case of the *Aymara*. We owe this major discovery to Iván Guzmán de Rojas [39], which deserves further consideration. From the first studies that were made of the language, their unique use of suffixes drew attention. Ludovico Bertonio called it the "machinery of particles". However, the many aspects of this linguistic apparatus were not studied in depth, especially with regards to its logic.

If we assume that logical thought must be expressed through language, one of the most important results to be derived from linguistics is the logical structure of spontaneous thought. We have already insisted on this fact. In the case of the *Aymara* language, the result is surprising.

Using powerful arguments, the cited study shows that Bertonio's "machinery of particles" expresses a logic that matches Lukasiewicz's modal logic. Furthermore, the *Aymara* people, with the aim of ensuring ongoing communications with the conqueror, adapted the Spanish language so it could express the different necessary logical values. We will analyze some cases to show this.

There are two modes of the statement that contain a different logical meaning. These modes are expressed by means of suffixes in *Aymara* or, in Spanish, by means of special idiomatic forms. One form of the statement, which expresses that $x$ is true, is:

$x$.pi = $x$ *pues*

By contrast, with this other type of statement:

$x$.ki = $x$ *nomás*

the idea that there exists the possibility of $x$, but not the certainty, is expressed.

In many regions of Spanish-speaking America there is a difference between "*ahora*" and "*ahorita*"–or the more emphatic "*ahorita nomás*". "*Ahora*" is affirmative; it conveys a certainty, a stated truth, whereas "*ahorita*" is only a possibility. Perhaps the diminutive is the way in which the peculiar Amerindian logic can translate into Spanish that which is explicit in *Aymara*.

According to the author, the miscommunication between conquerors and conquered originates in these examples. Lukasiewicz's entire modal logic is also found here. The different logical paths of the *Aymara* language are visited throughout the work, and a surprising natural dialectics–unknown until today–are elaborated with regards to the handling of opposites and specificities of the negation function. This leads to acquiring the conviction that this natural logic is not an extension of classical Greek logic but a different approach to the knowledge of the universe. The successive references to the *Aymara* logic reinforce this assertion..

In [39] we find an insistence that the *Aymara* logic has an algebraic ring structure and no importance is given to its character as a lattice. In contrast, this work insists on its nature as a lattice and the partial order is identified as the essential property of dialectics.

## Dialectics in quantum mechanics

In this section we will analyze von Neumann and Birkhoff's quantum logics. The need to introduce a new logic to interpret quantum mechanics arises from the formulation of propositions regarding the electron's spin. The behavior of the spin is unique and can be consulted in the bibliography, see [48, 49] for more details. In summary, this behavior can be expressed by the diagram which appears in Figure 6. *Spin X+* is used to designate an electron with the spin precisely defined and directed towards the positive X-values. Analogously, *spin X-* is defined when directed towards the negative X-values and the corresponding cases for the Y axis.

In quantum logic, the non-distributive property of this lattice is used to formulate the following non-equivalent statements:

*spin X+* **AND** (*spin Y+* **OR** *spin Y–*)

(*spin X+* **AND** *spin Y+*) **OR** (*spin X+* **AND** *spin Y–*)

The first statement is true in the physical world, whereas the second is always false since it is not possible to simultaneously specify the spin in two different directions. The fact that the logic of a particle's spin is *non-distributive* can be drawn from this. This is as far as physics go.

It is interesting to observe that the state of elementary particles leads to certain logic. This idea, which we have encountered on other occasions, also appears in the physical world. Secondly, it is interesting to note that the value of the first statement, which is true, is not 1 but "*spin X+*", it is a partial and not an absolute truth. It is also interesting to note that the environment of quantum logic is a **D4** dialectic lattice as it appears in Figure 6.

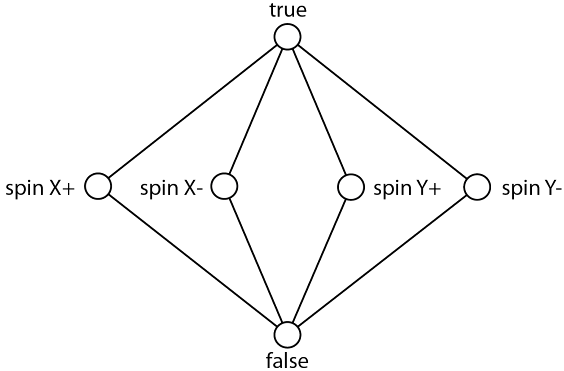

Figure 6: Quantum spin, **D4** dialectics.

Finally, there is a natural negation in this logic–a notion which has not been handled by traditional quantum logic–, which is the following:

$$N = (0\ 1)(\textit{spin X+}\quad \textit{spin X–})(\textit{spin Y+}\quad \textit{spin Y–})$$

which establishes relations of inversion of the particle's spin.[76]

## The dialectics of Pythagoras and his heirs

Pythagoras (–569?, –475?) has been studied mostly by Diogenes Laërtius ($3^{th}$ century B.C.).[77]

[76] This is not the time to develop Niels Bohr's notion of complementarity which is linked to the study of negations in quantum lattices and dialectic opposites.

[77] I leave out the quote on the straight-angle triangle because it bears no relation to the topic at hand.

> *The principle of all things is the monad or unit; arising from this monad the undefined dyad* [ ... ] *the elements of which are four, fire, water, earth and air; these elements interchange and turn into one another completely, and combine* [ ... ] *There are also antipodes, and our "down" is their "up". Light and darkness have equal part in the universe, so have hot and cold, and dry and moist* [ ... ] [17, VIII, 25–26]

> [*Living creatures*] *It has in it all the relations constituting life, and these, forming a continuous series, keep it together according to the ratios of harmony, each appearing at regulated intervals.* [17, VIII, 29]

> *The soul of man, he says, is divided into three parts, intelligence, reason, and passion.* [17, VIII, 30]

In these fragments we find several dialectic notions: the idea of opposites, of "harmonious" relations–which were subsequently attributed to music–and the triple structure of human nature.

Multiple stories involving his musical theory and its relationship to the celestial spheres can be added to this information. The idea behind this is simple: the moving spheres, visible to the naked eye, are 7–the Sun, the Moon, Mercury, Venus, Mars, Jupiter and Saturn–and so is the musical 7-note scale.[78] This led to the correspondence between these two structures–"ordered" structures, according to the medieval idea–as an essential part of the universe being attributed to him. The rotation of the celestial spheres would be responsible for "heavenly music".

The ideas attributed to Pythagoras have persisted throughout the centuries. They have appeared many times at several moments and with regards to various topics. From all of these we will analyze two cases: Paracelsus and Kepler.

Paracelsus is the *nom de plume* of philosopher, doctor and alchemist Theophrastus Bombastus von Hohenheim (1493, 1541). It is interesting for us to go over his main ideas which describe his vision of the

[78] The Greeks had nothing like the 7-note scale, see [33], this is a medieval, and not Greek, notion. It is one of the many legends on Pythagoras.

universe, see [73].[79] The first statement is the existence of the three "ingredients" of the universe:

> *He* [God] *originally reduced it to one body, while the elements were developing. This body He made up of three ingredients, Mercury, Sulphur, and Salt, so that these three should constitute one body. Of these three are composed all the things which are, or are produced, in the four elements.* [73, GE, I, vi]

We must first and foremost point out that Mercury, Sulfur or Salt are not the ordinary products known as such, but something more akin to the "Platonic" or "philosophical" essence of these products. From this basic triad, the quartet of Greek elements is generated, and from there, the seven metals.

> *From that chaos God built the Greater World, separated into four distinct elements. Fire, Air, Water, Earth. Fire was the warm part. Air only the cold, Water the moist, and, lastly, Earth was but the dry part of the Greater World.* [ ... ] *If this, by alchemical art, be anatomised and separated, all the seven metals, and these pure and unmixed, proceed from it, namely, gold, silver, copper, tin, lead, iron, quicksilver* [ ... ] [73, NT, VIII]

Also, these metals are "philosophical" and not vulgar, as the following passage from the *Catechism* shows, and as appears in other passages from his books:

> *Q. When the Philosophers speak of gold and silver, from which they extract their matter, are we to suppose that they refer to the vulgar gold and silver?*
>
> *A. By no means; vulgar silver and gold are dead, while those of the Philosophers are full of life.* [73, CA]

[79] This book contains a compilation of Paracelsus' works. The following abbreviations will be used with the aim of condensing the references: *The Coelum Philosophorum* (CP), The Aurora of the Philosophers (AP), The Generations of the Elements (GE), Concerning the Nature of Things (NT), A Short Catechism of Alchemy (CA).

The "philosophical" metals are identified with the seven moving spheres, which, in turn, are not visible, but "philosophical" objects. The correspondence is as follows: Sun ↔ Gold, Moon ↔ Silver, Mercury ↔ Mercury, Venus ↔ Copper, Mars ↔ Iron, Jupiter ↔ Tin y Saturn ↔ Lead. These metals can be transmuted between each other following some very precise rules.

> [ ... ] *pay attention to Saturn, which is the highest of all, and then is succeeded by Jupiter, next by Mars, the Sun, Venus, Mercury, and, lastly, by the Moon.* [ ... ] *experience teaches us that Mars can be easily converted into Venus, not Venus into Mars, which is of a lower sphere. So, also, Jupiter can be easily transmuted into Mercury, because Jupiter is superior to Mercury, the one being second after the firmament, the other second above the Earth, and Saturn is highest of all, while the Moon is lowest. The Sun enters into all, but it is never ameliorated by its inferiors.* [73, CA]
>
> [ ... ] *in order to transmute the five lower and baser metals, Venus, Jupiter, Saturn, Mars, and Mercury, into the two perfect metals, Sol and Luna, you must have the Philosophers Stone.* [73, NT, VII]

Ultimately, the chain of possible transmutations is:[80]

> Saturn (lead) → Jupiter (tin) → Mars (iron) → Venus (copper) ⇒ Moon (silver) ⇒ Sun (gold)

where → indicates a simple transmutation or from greatest to lowest, and ⇒ indicates a transmutation that implies the *philosopher's stone*.

Ultimately, Paracelsus' theory consists in a collection of universal correspondences which carry everything that is real to the seven planets-metals, which leads to the four Greek elements, the three "philosophical" ingredients and also the binary, female-male classification. This correspondence extends to the human body: Sulphur is likened

[80] It might be of interest to note that the chain of transmutations increases the value of metal until reaching its maximum for gold.

with our emotions and desires, Salt, with the body, and Mercury, with our higher mental faculties.

A century after Paracelsus, Johannes Kepler revisited Pythagorean thought, now applied to astrology. In 1618's *Armonice mundi*, he is preoccupied with the arrangement of the planets within the Solar System. The distances of the 5 planets to the Sun became a matter of obsession for Kepler, who went as far as to link them to the regular solids and the musical scales. He thus resumed Pythagoras' idea. As a sample of his results, we can quote the following statement:

> The extreme movements of the planets had to designate pitches or strings of the octave system, or notes of the musical scale. [50, V, proposición xxii]

What do Pythagoras, Paracelsus and Kepler have in common? In all of them we find a search for a universal correspondence between reality and a very simple mathematical or formal structure. These structures have 2, 3, 4, 5 or 7 elements, a direction of rotation and are the "ultimate representation" of reality. In the language used in this book, a correspondence with a lattice which admits a rotation is established and there is a form of becoming of the elements.

## Many-valued logic

This study intends to be a formal inquiry on logic. From this point of view, it may be considered an incursion into the subject of *many-valued logic*. It is well-known that this topic has been studied many times as an abstract generalization of Boolean logic. This work is different in that it attempts to focus on the issue of dialectics, which is why it ends with many-valued logic. Up to now, the approach of many-valued logic has always ended where it began. The words of Garrett Birkhoff are very enlightening in this respect:

> *Most systems of modes studied in the past have been simply ordered by the degree of truth which they ascribe to propositions. All other knows to me have formed distributive lattices*

> *and hence sub direct unions to two–valued logics. The author can see no valid reason for this emphasis on simple ordering. It would seem worthwhile to construct propositional calculi base on non–distributive lattices of truth–values –say, on the two non–distributive lattices of five elements. In my own attempts to do this, I have been troubled by the problem as to how the truth–values of* $p$ *and* $q$ *should determine the truth–value of* $p \Rightarrow q$ *and* $\neg p$. [4, XII, 8]

This situation springs from a reasonable attempt–using his mathematical intuition, Birkhoff sensed that the problem lay with **D3**, five-element lattices, the Hegelian lattice–but without an actual orientation to study the problem. Something similar happened to those who came after him: they did not stray from the idea that logic is built from negation and implication, but never considered new operations. As we will see in the chapters that follow, the issue is far from simple.

In this study, the problem is stated in reverse: dialectics exist in nature and it is therefore necessary to find their formal expression. As a consequence, we arrive at many-valued logic. So far, everything is clear. But our problem does not end with its mere formalization.

# An intuitive introduction to dialectics

## Introduction

This chapter makes an intuitive introduction of several notions. Its purpose is to make the transition between examples taken from the natural language, philosophical notions from the past and the formalization of dialectics as an abstract theory.

Throughout the previous chapter, the different figures have presented lattices without previously defining them. The various diagrams showed circles connected by lines. The underlying idea is that if an element (a circle) is connected to another, a relation of order is established between the upper and the lower element. In this way, for example, the elements *water* or *air*, as they appear in Figure 4 are inferior to the *moist* element, which is, in turn, inferior to the *all* element.

This structure, which connects elements to one another by means of an order or hierarchy, is what mathematicians call a *lattice*. We will introduce a formal definition in the next chapter.

This idea of order or hierarchy relates to the notion of "greater logical value than" which has appeared in the introductory chapters. This is the basis for the mathematical structure of a lattice, as we will see in the formal definition.

The various applications of natural dialectics strongly suggest that the different dialectic values are *formally equivalent*: nothing *formal* sets them apart. It is the semantics of the applications that which allows us to distinguish *yin* from *yang*, as occurs with the Chinese philosophers or Toynbee. This is not the case in Oscar Wilde. By the same token, among Ionian or classical Chinese elements, only semantics permit telling one from the other. What is more, East and West take into account different collections of elements, and even a different amount. The essential part of their thought relies on their existence and the rotation that occurs between the elements.

## Logic and lattices

The close connection that exists between logic and the algebraic structures known as *lattices* has been explained many times. Without intending to make an exhaustive list, we can remember the following cases:

- Boolean logics, see [4],
- many-valued logics, see [57, 58],
- Piaget's epistemological genetics, see [2, 76],
- quantum logic, see [48, 49],
- many-valued logics of technical use, see [90].

The links between lattices and Boolean logics are well-known. The first attempts at building many-valued logics, made by Jan Lukasiewicz (1878, 1956) and Alfred Tarski (1901, 1983) [57, 58] or Emil Post (1897, 1954)[82], led to very simple lattices, with linear chain structures. However, we should not overlook the fact that the formalization of lattice theory did not occur until 1933 to 1937, quite some time after these first attempts at a generalization. Possibly due to their simplicity, these attempts did not make as much progress as they could have.

In the study of the genesis of knowledge that Jean Piaget (1896, 1980) embarked upon, he encountered the notion of *lattice* many times, which led him to consider intermediate structures, the *groupement*,[81] between groups and lattices, as methods of expression of this genesis [75]. Ever since the beginning of the formulation of quantum mechanics, John von Neumann (1903, 1957), Garret Birkhoff (1911, 1996) and other authors [48] acknowledged the need for expressing the logical relations which occur in some aspects of the theory using more complex lattices than Boolean lattices. In this specific case, the distributive property of the logical operation **AND** suggested taking this path. However, to this day, these attempts have not yielded new results.

The modern developments in microelectronics have naturally led to many-valued logics with a direct technical application: see [48, 56, 90].

[81] A *formalization* somewhere of these ideas is presented in [2, III,1].

This background naturally leads us to consider lattices as the quintessential environment of logic. This fact definitely influences the existence of the logical operations **AND** and **OR** as basic lattice operations. However, in order to introduce logic into a lattice–and even more so, in order to formalize dialectic logic–, it is necessary that we specialize lattices and introduce new notions. This work intends to present "the general environment of logic" and give precise form–or better yet, algebraic form–to the statements belonging to spontaneous human thought, all the way to the ideas of the classics of dialectic materialism.

## Lattice operations

In a lattice, two dual operations between two elements can be defined. One operation connects each pair of elements to a third one which is "superior" or higher up in the diagram, yet still connected to both. This operation is called **OR**. In a dual manner, **AND** is defined as the "inferior" to the two elements. Therefore, for example, the definitions that are found in page 60 can be complemented as follows:

| | |
|---|---|
| *water* = *moist* **AND** *cold* | *water* **OR** *air* = *moist* |
| *earth* = *cold* **AND** *dry* | *earth* **OR** *water* = *cold* |
| *fire* = *dry* **AND** *hot* | *fire* **OR** *earth* = *dry* |
| *air* = *hot* **AND** *moist* | *air* **OR** *fire* = *hot* |

The idea is to express, for example, that *water* is what *moist* and *cold* have in common, and that the result of the union of *water* and *air* is *moist*. These two operations, dual between themselves, have the same formal structure as the **AND**, **OR** operations that belong to the realm of logic.

Lattices have a maximum and a minimum element. In some diagrams, "true" and "false", "all" and "nothing" or other appropriate terms have been used. From now on, 1 and 0 will also be used.

The remaining notations and necessary operations will be defined as they are introduced in the formal exposition. As a general reference to lattices, please refer to Birkhoff's classical books [4, 5] or his most recent, [16].

## Semantics of logical values

From early on in this work, we assumed that there was a direct correspondence between the uppermost value or supremum of the lattice, which we referred to as 1, and the logical value "true". In much the same way, the logical value "false" was taken as 0, to match the lowest value or infimum of the lattice. The intermediate elements, the dialectic values, are a novelty introduced by dialectics..

*Dialectic* values are intermediate logical values between "true" and "false", which we call "theses" by extension of Hegel's terminology. Generally speaking, we can say they represent intermediate degrees of truthfulness or falseness. Further ahead we will offer examples of interpretation for these cases. Historically, in modal logic, Lukasiewicz [58] referred to the intermediate value of **C3** as "hypothetical", while Reichenbach [49] called it "indeterminate". In technical logic, different values for "true" and "false" were introduced using other denominations.

Traditional propositional calculus can therefore be generalized right away through the following definition:

**Definition 1** *We will refer to "thesis value"–or simply, "thesis"–as any value different from 0. We will refer to as "true" or "absolute truth" as the value 1, and "false" or "falseness" as the value 0.*

This definition differs from Lukasiewicz' interpretation, who though– in some way, in degrees of truthfulness–that intermediate values were doubtful. In dialectics, they express values which are not absolutely truthful, either because they are stages of a possibly incomplete knowledge, because they may change, or because they are intermediate stages in a process of becoming. Therefore, in the examples of natural dialectics we have already analyzed, the following, among others, are theses:

- Artistic truth, as Oscar Wilde put it. Although not an absolute truth, its opposite is also true and not false–it is an intermediate value.
- Statements which make use of the conjunction "but". They state

something which is less than **OR** but more than **AND**.

- Transient states of becoming, such as the *yin* or *yang* states in the dialectic of Toynbee's history.

It is necessary to distinguish some additional types of propositions which do not exist in binary logic. An expression may be of various types, as shown in the following definition.

**Definition 2** *A dialectic function of several variables, depending on the logical value it acquires for the different values of its variables, is called:*

- *a* tautology *or* truth *or* absolute truth *if it is always worth 1,*
- *a* falseness *if it is always worth 0,*
- *a* thesis *if it is never worth 0,*
- *a* strict thesis *if it only acquires dialectic values.*

We can generalize the terminology of traditional binary logic in this way.

## The science of logic

We will refer to as *logic*, without any qualifiers, or *binary logic*, as the structures which occur within a binary lattice made up of two elements, designated as **B**. The natural dialectics examples we presented had special lattice structures. One of our primary concerns will be to establish boundaries for the lattices in which dialectic logic takes place. This is by no means a minor endeavor. We will refer to these non-binary lattices as *dialectic lattices*. Of course, while this is only an informal and imprecise definition, it simply intendst to introduce the terminology.

The systematic study of the properties of dialectic lattices is a specialized area of algebra which can be referred to as *formal dialectics*, or, better yet, the *science of logic*. Throughout this study we will only analyze a specific class of lattice. We hope that this set is comprehensive enough to include the main aspects of the issues that concern thought.

Intuitively speaking, the dialectic logic in this inquiry is a "quantified" modal logic. Between the values 0 (false) and 1 (true) there is a finite and discrete amount of values. Unlike Lukasiewicz' chains, there are several elements which may have equivalent logical value. This double condition of modal logic and multiple equivalent values is what determines the richness of the proposed dialectic logic.

# The formalization of dialectics

## Dialectic logic as an image of the universe

The dialectic examples analyzed in the previous pages force us to reconsider the notions of traditional logic. In some way, logic appears as a structure associated to the fundamental properties of the universe, as the bearer of supreme laws of matter and movement. This is the thesis traditionally defended by dialectic materialism.

Since the study of the universe is based on the formulation of statements, a connection between statements constitutes knowledge of the universe. The basic structures of the universe must somehow correspond to structures within these statements.

As a summary, dialectic logic can be defined as a combination of three basic ideas:

- an *order relation*, which establishes a hierarchy of "logical value" of the statements,
- a *homomorphism*, which maintains and acts as a reflection for the relationships between statements,
- an intrinsic *circularity* within this structure.

The order relationship is a much generalized, intuitive idea, which can be expressed in several ways: "it is truer than" or "my reasons are stronger than"–something usual within the context of a discussion–, "these ideas are simpler than"–Occam's razor–, or spontaneous deductive reasoning: "this is the result of that". Dialectic logic must provide an answer to this order relationship.

Dialectic logic appears as a structural correspondence–a *homomorphism*, a mathematical notion which will be defined with precision further ahead–, which in turn establishes a correspondence between statements capable of expressing knowledge of the universe, within a condensed, simpler structure, made up of a small number of elements: three, four, five, twelve or more. The existence of this homomorphism accounts for the coincidence between states or properties of matter and the logical structures capable of "explaining" these properties.

The idea of circularity is present in this correspondence: the *four Western elements*, the *five Chinese elements* or the *three* universal entities–*yin*, *yang*, synthesis–, magical–numerological or astrological–interpretations, and so many others, present some degree of circularity. They are budding homomorphisms, although based on reasonably accurate attempts. Pythagoras' thought–and that of his followers–consists in a homomorphism of *seven elements* comprising a certain order, and hence the importance of the moving celestial objects, the musical scales and many other similar structures. Astrology–both Western and Eastern–is nothing but a speculation on *twelve elements* inspired by the twelve moons of the year. Their periodical nature is self-evident.

A combination of the three basic logical properties naturally leads to the algebraic structure of a lattice containing specific properties–to be defined later on–, that we will consider *dialectic* in nature since it presents the three properties mentioned above.

Lattices and logical values materialize these material properties into images, hence this unexpected coincidence between "elements" and logical values. When the spontaneous epistemology of humans discovered homomorphism, he tried to dress it as something abstract, with varying degrees of success. In fact, George Boole [6] was the first to achieve a formalization. This homomorphism on algebraic structures reflects knowledge of matter and change, and shows that both logic and mathematics are not the unrestrained creation of the human brain, but something tied to our material universe with the strength of a law of nature.

## Lattices overview

Lattices are algebraic structures–either finite or infinite–that are defined based on an elementary notion, that of *partial order*. This notion is defined as follows.

**Definition 3** *A set, **L**, is said to be* partially ordered*–or simply* ordered*–if a relation $\leq$, is defined among the elements of **L**, which, for every $x, y, z \in$ **L** meet the following:*

1. *Reflexive or idempotent (I): $x \leq x$.*
2. *Antisymmetrical: if $x \leq y$ and $y \leq x$, then $x = y$.*
3. *Transitive (T): if $x \leq y$ and $y \leq z$, then $x \leq z$.*

By extension, we can say that $x < y$ if $x \leq y$ but $x \neq y$.

The simplest example is the *chain*, a linearly ordered set: given two elements, $x, y$, either $x \leq y$ or $y \leq x$. All the elements are comparable among themselves. A more complex example is **2D4** in Figure 4. In this case, two elements are comparable if there is a line that joins them in the diagram. Thus, for example, we have: *false* $\leq$ *earth* $\leq$ *dry* $\leq$ *true*. By contrast, *earth* and *fire* are not comparable to each other but have *dry* as greater than both.

**Definition 4** *A set, **L**, is said to be a* lattice *if for $x, y, z \in$ **L**:*

1. *It is a set partially ordered by the relation $\leq$.*
2. *For every pair of elements $x, y$, there is an element, $x \,.\, y$ which is the greatest of all the elements inferior to both, that is, if $z \leq x$ and $z \leq y$, then $z \leq x \,.\, y$.*
3. *For every pair of elements $x, y$, there is an element $x + y$ which is the smallest of the elements greater than both, that is, if $x \leq z$ and $y \leq z$, then $x + y \leq z$.*
4. *There is a* top element, *which we will call 1, and a* bottom element, *which we will call 0.*

It follows that the operations defined in the lattice are idempotent (I), associative (A) and commutative (C). Therefore, the operations may extend to several elements. Due to an abuse of the language, the two operations will be referred to as *sum* (+) and *product* (.).

For every lattice, we can define the idea of contiguous elements.

**Definition 5** *Two elements $x, y$ of a lattice are considered* contiguous *if $x < y$, and there is no element $z$ in the lattice that meets $x < z < y$.*

Some lattice elements have a specific name.

**Definition 6** *In every finite lattice, elements contiguous to 1 are referred to as* maximum *elements, elements contiguous to 0 are called* atoms.

In this book we use the *technical notation* for lattice operations. Therefore, as an example, the following occurs in **2D4**, Figure 4:

*dry . cold = earth*
*air + water = moist.*

In the *mathematical literature on lattices*, see [4, 5], ∩ is used for the dot and ∪ for the + sign.[82] The example would be as follows:

*dry ∩ cold = earth*
*air ∪ water = moist.*

The logical notation can also be used: the dot is read as **AND**, and the + sign as **OR**.[83] The example would be as follows:

*dry* **AND** *cold = earth*
*air* **OR** *water = moist.*

The monotony properties of the lattice operations are important and can be found in the following theorem.

---

[82] In [16], the symbols ∨ y ∧ are used, respectively, in lieu of ∪ y ∩, in order to avoid confusion with the union and intersection operations for the set.

[83] Russel and many other logicians use ∨ –which is reminiscent of the Latin word "*vel*", which expresses the disjunction– for **OR** and the dot for **AND**, see [86].

**Theorem 1** *If* $x, y_1, y_2, \ldots, y_p, z_1, z_2, \ldots, z_q$ *are elements in a lattice, then if* $x \leq y_i$ *y* $z_j \leq x$ *then*

$$x \leq y_1 . y_2 . \cdots . y_p \qquad x \leq y_1 + y_2 + \cdots + y_p$$

$$z_1 . z_2 . \cdots . z_q \leq x \qquad z_1 + z_2 + \cdots + z_q \leq x$$

*which are the* monotony *properties of the operations.*

**Proof**. It is clear that $x \leq y_1 . y_2$ since $y_1 . y_2$ is the maximum lower bound of $y_1, y_2$ and $x$ must be equal or lesser than this bound. Applying this same result to $y_1 . y_2$ and $y_3$ the following factor is added and so forth. With this, the first property is proven. The second property immediately follows, given that $x \leq y_1 + y_2$ because $x \leq y_1 \leq y_1 + y_2$. By repeatedly applying this result, it is proven. The third property immediately follows, given that $z_1 . z_2 \leq z_1 \leq x$. By repeatedly applying this result, it is proven. $z_1 + z_2 \leq x$ given that $z_1 + z_2$ is the minimum upper bound of $z_1, z_2$, then $x$ is equal or greater. By repeatedly applying this result, as in the first case, the fourth property is proven. □

Several correspondences between elements can be defined between two lattices:

**Definition 7** *A* homomorphism ***H*** *between two lattices is referred to as a correspondence between* $x_i \in$ **L** *and* $y_i \in$ **M** *such that* $x_1 . x_2$ *and* $x_1 + x_2$ *belonging to* **L** *respectively correspond to the elements,* $y_1 . y_2$ *and* $y_1 + y_2$ *belonging to* **M**. *In other words, the two operations between lattice elements are preserved. If Si* **L** = **M**, *it is called an* automorphism. *If the correspondence is biunivocal between* **L** *and* **M**, *it is called an* isomorphism. *If the corresponding elements of* $x_1 . x_2$ *and* $x_1 + x_2$ *are, respectively,* $y_1 + y_2$ *and* $y_1 . y_2$, *it is called a* dual *or* reverse isomorphism *or* antiisomorphism.

In this study there is an important case of homomorphism, which we will refer to as a *reduction homomorphism* or *R-homomorphism*.

**Definition 8** *An* R-homomorphism *or* reduction homomorphism *between lattices* **L** *and* **M***, is a homomorphism such that* **M** *has* fewer elements *than* **L***. A lattice lacking a nontrivial* R-homomorphism *is referred to as an* irreducible lattice.

Intuitively, an irreducible lattice cannot be equated with a simpler one and still maintain its logical properties. Thus, for instance, the *yin-yang* lattice in Figure 1 has an R-homomorphism, $\mathbf{H}_r$, given by the following correspondences:

$\mathbf{H}_r$: 0, *yin* $\rightarrow$ 0'   $\mathbf{H}_r$: *yang*, 1 $\rightarrow$ 1'.

This R-homomorphism transforms the *yin-yang* lattice into the binary logic lattice 0', 1 while maintaining the operations $\cdot$ and $+$. The *yin-yang* lattice is reducible. By contrast, the Hegelian lattice in 2 is irreducible. Figure 7 introduces a lattice with an R-homomorphism.

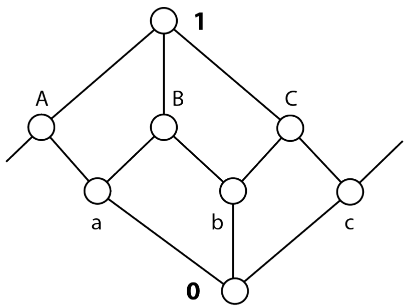

Figure 7: The **2D3** lattice as an example of homomorphism.

This homomorphism, $\mathbf{H}_r$, is given by the following correspondences:

$\mathbf{H}_r$: $0, b, c, C \rightarrow 0'$   $\mathbf{H}_r$: $a, A, B, 1 \rightarrow 1'$.

As an example, $1 = A + B \rightarrow A' + B' = 1'$, $a = A \,.\, B \rightarrow A' \,.\, B' = 1'$ and so on for the remaining possible cases. The general study of homomorphisms in dialectic lattices falls outside the scope of this study.

Given two lattices, the *direct product*, *Cartesian product* or simply, the *product* of these lattices can be defined as:

**Definition 9** *Given $s$ lattices $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_s$ the direct (or Cartesian) product $\boldsymbol{L}_1 \times \cdots \times \boldsymbol{L}_s$ is defined as the lattice made up of the elements $(a_1, \cdots, a_s)$, a set of elements of the $s$ lattices, by means of the order relation:*

$$(a_1, \cdots, a_s) \leq (b_1, \cdots, b_s)$$

*if, for every $i$, $a_i \leq b_i$ is met for the elements of each $\boldsymbol{L}_i$.*

The most well-known case of this product occurs with the binary lattice of two elements, the Boolean logic, see Figure 11. In [16] it is established that all Boolean logic of a finite number of elements is a $\mathbf{B}^n$ power of the simple Boolean logic.

The product of lattices is not of much interest in dialectics. It is clear that the Cartesian product, for example, of $\mathbf{L}_1 \times \mathbf{L}_2$ is homomorphic both to $\mathbf{L}_1$ and $\mathbf{L}_2$. Homomorphisms are very simple: $\mathbf{H}_r$: $(a_1, a_2) \to a_1$ and likewise for index 2.

## Some lattices of logical interest

One specific set of lattices and functions is of special interest to this work. The following definitions introduce these cases. With the aim of completing the notation, **Bn** or $\mathbf{B}^n = \mathbf{B} \times \mathbf{B} \times \cdots \times \mathbf{B}$ will refer to the n-th power of the Boolean lattice made up of the direct product of $n$ **B** Boolean lattices.

A Lukasiewicz-Post lattice, **Cn**, containing $n$ elements, refers to the chain of $n$ elements between 0 and 1. The elements have the following values:

$$\frac{p}{n-1} \quad \text{where} \quad p = 0, 1. \cdots, n-1.$$

These are rational $n$ values between 0 and 1. In this structure, a logical negation, **N** can be defined, which has the following property:

$$\mathbf{N}\ \frac{p}{n-1} = \frac{n-1-p}{n-1}.$$

This logic, thus constructed, is defined in [57, 58, 82]. In this logic, Lukasiewicz includes two modal functions: *certainty* (*Gewissheit*) and *possibility* (*Möglichkeit*).[84] These aggregates fail to thoroughly illustrate what happens in natural dialectics or the proposals of the present work.

In the ternary logic of Hans Reichenbach (1891, 1953) [48, 49] three functions referred to as negations are considered in the **C3** lattice. The "cyclical" function exchanges all the elements and is similar to the first of the negation functions introduced by Post [82] in **Cn**. Evidently, Augustus De Morgan's (1806, 1871) property is not met. Reichenbach's "complete negation" lacks an inverse function and does not comply with De Morgan either. Finally, the "diametrical negation" is a negation as per Lukasiewicz's definition. Both Lukasiewicz [57] and Post [82], in their second definition, match Lukasiewicz's definition of negation in **Cn**.

## Dialectic lattices

Loosely defined, a dialectic lattice has two basic properties:

- A *rotation* which transforms it into itself, an *automorphism*.
- An *antiisomorphism* which transforms $x + y$ into $x' \,.\, y'$ and dually, where $x', y'$ designate the elements corresponding to $x, y$.

Irreducible lattices are the most important set of dialectic lattices. However, reducible lattices are also of interest in some cases. For this reason, the definition of a dialectic lattice is not required to have this property.

Let us consider lattices which generate a dialectic, and not a Boolean, logic. We will begin by posing the general definition of a dialectic lattice, resulting in the subsequent definitions.

> **Definition 10** *A* dialectic lattice *of* rank 1 *and* size $n$, ***Dn****, is referred to as the lattice made up of bounds* 0 *and* 1 *and* $n$ *atoms,* $d_i$*, which meet* $0 < d_i < 1$*, and are called* dialectic elements.

[84] If $d$ designates an intermediate value between 0 and 1, Lukasiewicz respectively defines as $G(0) = 0, G(d) = 0, G(1) = 1$ and $M(0) = 0, M(d) = 1, M(1) = 1$, *Gewissheit* and *Möglichkeit*.

These lattices are of major importance to this work. By definition, the following cases occur:

**D0** = **B** = **C2**, dialectic lattice of size 0, defined, by an abuse of the language, as matching the simple or binary Boolean lattice, or as the minimal Lukasiewicz chain.

**D1** = **C3**, matches Lukasiewicz' chain of size 3.

**D2** = $\mathbf{B}^2$, matches the Boolean lattice of size 2 and the *yin–yang* dialectics, Figure 1.

**D3**, Hegel's dialectic lattice, the basic case in dialectic logic, Figure 2.

**D4**, dialectic lattice of size 4, matches the Ionian elements, Figure 3.

**D5**, dialectic lattice of size 5, matches the Chinese elements, Figure 5.

The **2D4** lattice, Figure 4, is comprised within a more complex series of dialectic lattices. This situation can immediately be extended to a more general case.

**Definition 11** *A* dialectic lattice *of rank $r$ and size $n > r$, **rDn**, is referred to as the lattice made up of the bounds* 0 *and* 1 *and $r$ sets of $n$ elements 0 $< d_{i,j} <$ 1, referred to as dialectic. An orden relation such that $d_{i,j} < d_{i+1,j-1}$ y $d_{i,j} < d_{i+1,j}$ is met. With $i = 1, \cdots, r$ and indexes $j = 0, \cdots, n-1$, they are considered as $n$-module. Those that meet the conditions of Theorem 4 are not considered lattices.*

According to this definition, **Dn** lattices may also be designated as **1Dn**; using a shorter notation is simply preferable. On the other hand, these lattices have properties–given that atoms and maximum elements match–which make them different and call for a different demonstration of some theorems.

The definitions of dialectic lattices utilize the elements regarded as characteristic of dialectic logic. Because we are dealing with lattices, there is an orden relation between their elements.[85] Lattice **3D5** in Figure 8 intends to illustrate these notions. It follows that this lattice has n automorphisms which match the rotations of the $n$ atoms. It also presents the following properties.

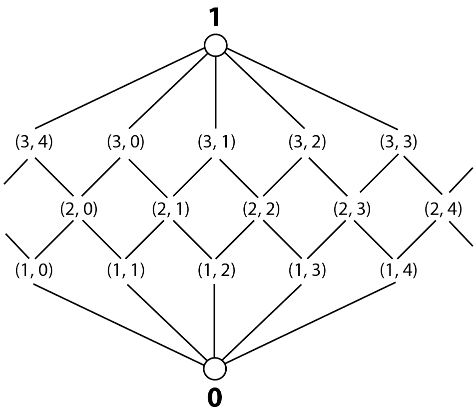


Figure 8: **3D5** dialectic as an example of the general case.

**Theorem 2** *Every dialectic lattice* ***rDn*** *has the following properties:*

- *It has $n$ atoms, $d_{1,j}$, and $n$ maximum, $d_{r,j}$. The product of the two atoms is* 0 *and the sum of the two maximums is* 1.
- *The sum $d_{i,j} + d_{i,j+1}$ of two contiguous elements is $d_{i+1,j}$ and the product $d_{i,j} \,.\, d_{i,j+1}$ of two contiguous elements is $d_{i-1,j+1}$, all the operations are $n$-module.*
- *It has a rotation defined as $R_1\, d_{i,j} = d_{i,j+1}$ (the sum is considered to be $n$-module) and all of the successive applications of this transformation are also rotations.*
- *Every element $i$, $j$ is equal to the sum of the atoms of the lattice which are smaller than it and the product of the maximums elements which are greater than it, in both cases the basic operations of the lattice are considered.*

[85] This definition can be generalized further by increasing the number of indexes: the first is the rank $r$ and the following are of size $n$. The operations are $n$ module.

**Proof**. These results follow from Definition 10. □

Dialectic lattices display a very simple property, as established by the following theorem.

> **Theorem 3** *In a lattice* ***rDn****,* $r > 1$*, every maximum element is the sum of r atoms. The necessary and sufficient condition for a maximum element* $D$ *and an atom* $d$ *to exist, such that* $D \,.\, d = 0$ *and* $D + d = 1$ *is that* $n > r$*. The property is also met in a dual manner, by exchanging maximum elements and atoms.*

**Proof**. Let us consider the atoms which are smaller than a maximum element $(r, 0)$, se Figure 8. It follows that they are: $(1, 0) \cdots (1, r - 1)$, $r$ atoms in total. For an atom and the maximum element to add up to 1, at least one additional atom must exist, which will only occur if si $n - 1 \geq r$ that is, if $n \geq r + 1$. These atoms also meet the condition for the product. The difference $n - r$ is the number of atoms that meet this property and, therefore, they are not smaller than the considered maximum. The dual case is proven in the same manner. □

> **Theorem 4** ***rDn*** *structures, with an even* $r$*, are not lattices if they meet* $2(r - 1) \geq n$*.*

**Proof**. In order to establish these concepts, we will consider Figure 9 with the **3D4** structure. We can immediately detect that there are two values for the sum $a + c = A, C$, for example, which is unacceptable in a lattice. Dually, there are also two values for the product $B \,.\, D = b, d$, for instance. In general, this occurs in structures with an even $n$, and where there are two sets of $r$ atoms which complete the period $n$ of the sum of the corresponding indexes, that is, the condition $2(r - 1) = n$. If we take **4D6**, $r$ consecutive atoms add up to a maximum and the same case of the double result of the sum appears once again. What is more, this also happens in **5D6**, where not only 4 atoms have a double sum, they also have 4 elements of logical level 2, see Definition 12, and

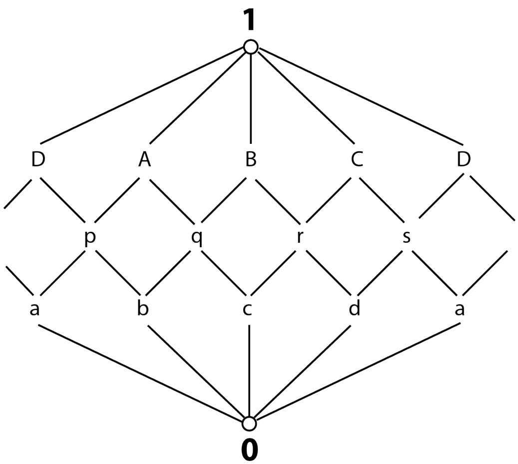


Figure 9: **3D4** structure as an example of a non-lattice.

so do the corresponding products, in a dual manner. Therefore, for every $r$ such that $2(r-1) \geq n$ as long as $r < n$ the double sums and products appear, with no lattice involved. Table 3 presents structures *which are not lattices*. □

Table 3: Examples of non-lattices.

| n | r | cases |
|---|---|---|
| 4 | 3 | **3D4** |
| 6 | 4 | **4D6**, **5D6** |
| 8 | 5 | **5D8**, **6D8**, **7D8** |
| 10 | 6 | **6D10**, **7D10**, **8D10**, **9D10** |
| … | … | … |

These false lattice cases would yield a dialectic where two conclusions of the same logical level might be possible–see Definition 12–based on the same premises. This dialectic might be of interest and be applicable to certain scientific purposes, but the present study will not focus on this.

A major notion in dialectics is the idea of *logical level* of an element that is related to the element's "degree of truth" or proximity to 1.

**Definition 12** *The number $i$ refers to the* logical level *of an element $d_{i,j}$ of lattice **rDn** –generically called dialectic elements of the lattice. The dialectic elements of **Dn** have logical level 1.*

A basic theorem of automorphism derives from this definition.

**Theorem 5** *An automorphism transforms an element of a dialectic lattice into another element of* equal logical level. *This relation is an equivalence relation.*

**Proof**. The logical level $s$ of element $d_{s,t}$ allows to build a chain of the type $0 < d_{1,p} < d_{1,q} < \cdots < d_{s,t}$ with contiguous elements. Reciprocally, if the chain exists, the logical level is $s$. The automorphism transforms this chain into another with the same number of elements and the same relations, then it maintains the logical level. The automorphisms in a lattice make up a *group*. Then, $a = I\,a$, where $I$ is the identity automorphism. If $A$ is an automorphism and $b = A\,a$ then $a = A^{-1}\,b$. If $b = A_1\,a$ amd $c = A_2\,b$ then $c = A_2\,A_1\,a$. The three conditions for equivalence are met, idempotency (I), commutative (C) and transitive (T), therefore it is proven. □

**Theorem 6** *If two different elements in a dialectic lattice have the same logical level, then they are* not comparable.

**Proof**. Following the definition of a lattice, for two elements $a$ and $b$ to be comparable–that is, related to each other as $a \leq b$ or inversely–it is necessary for them to have a different first index, that is, logical level. □

**Definition 13** *An element $d_{s,i}$ of a lattice **rDn**, where $r = 2s - 1$, is referred to as the* central element *of the lattice.*

Central elements have a chain of $s$ elements up to 0 and also $s$ elements up to 1–that is why they are referred to as central. In Figure 8 elements (2, i) are central elements.[86] In the Hegelian example, the three elements $t, a, s$ are central values. By contrast, the lattice in Figure 4 has no central values.

The range $r$ of **rDn** lattices determines a certain type of dialectic. There is only one zero-range dialectic, and that is binary logic. There is infinite number of other dialectics depending on the number of atoms, but each rank defines a family that has different properties. Rank-1 dialectics can be referred to as Hegelian or simple and they serve to analyze the problems of contraries and becoming. Rank-2 dialectics allow us to analyze problems concerning historical materialism related to class struggle. Rank-3 dialectics allow us to analyze some of the problems linked to scientific theories and problems belonging to historical materialism related to the succession of the modes of production. It is possible that greater ranks may be useful in certain scientific or epistemological problems or those associated with the evolution of the species, but this study will not delve into logics belonging to these ranks.

The notion of complex lattices can be extended to dialectic lattices having three or more indexes–the first one refers to the logical level and the last one to the number of elements of equal level. It is possible that they may be useful in understanding the logic of scientific theories, but this topic is only suggested in this study.

The set of lattices defined comprises the cases of interest, as were mentioned in the beginning of this section. A two-dimensional diagram, Table 4, will allow us to properly visualize the mutual relations. Those that fulfill Theorem 4 are not in bold and therefore, they are not lattices.

This table associates the various lattices. As we move along the horizontal axis, the number of existing atoms and elements increases. If we move vertically, the rank increases and it is possible to have more logical values between "false" and "true", which we generically refer to

[86] If there is a negation $N$ such that the equation $N\,x = x$, then $x$ is a central element, as occurs with all the elements in **Dn**. As is clear, if a negation has this property then is not a *strict negation*.

Table 4: Dialectic lattices by rank and number of atoms.

| r / n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | … |
|---|---|---|---|---|---|---|---|---|
| 0 | **C2 = B=0D1** | | | | | | | |
| 1 | *C3 = 1D1* | **B$^2$ = 1D2** | **1D3** | **1D4** | **1D5** | **1D6** | **1D6** | … |
| 2 | *C4 = 2D1* | | **2D3** | **2D4** | **2D5** | **2D6** | **2D7** | … |
| 3 | *C5 = 3D1* | | | 3D4 | **3D5** | **3D6** | **3D7** | … |
| 4 | *C6 = 4D1* | | | | **4D5** | 4D6 | **4D7** | … |
| 5 | … | | | | | 5D6 | **5D7** | … |
| 6 | … | | | | | | **6D7** | … |

as *dialectic elements*. $rD\infty$ can also be defined. These are of interest for purposes of some endless processes, such as the case, for instance, of the river in Heraclitus, see page 58.

## Dialectic lattices and automorphisms

Automorphisms in dialectic lattices–except for Dn lattices–are made up of two families: rotations and symmetries. Rotations have been defined in Theorem 2, and the present section will deal with symmetries. In **Dn** lattices, any permutation of the elements is an automorphism.

In **2Dn** lattices the following theorem applies. We will use the mathematical notation which employs $d_i$ for atoms and $D_i$ for maximum elements. The following theorem shows that there are $2n$ automorphisms in the lattice.[87] There are $n$ rotations and an equal number of symmetries.

> **Theorem 7** $S_j$ symmetries *in* **2Dn** *fulfill the equations* $S_j\, d_i = d_{n-i+j}$ *and* $S_j\, D_i = D_{n-i+j-1}$, *the operations are* $n$*-module.*

**Proof**. It is only necessary to verify the properties for the sums of the atoms or the product of contiguous maximums elements. We will consider two contiguous atoms $d_i + d_{i+1} = D_i$, applying the symmetry we obtain $S_j(d_i + d_{i+1}) = S_j\, D_i = D_{n-(i+1)+j}$. But $S_j\, d_i = d_{n-i+j}$ and $S_j\, d_{i+1} = d_{n-(i+1)+j}$, then $S_j\, d_i + S_j\, d_{i+1} = d_{n-i+j} +$

[87] This result has been directly proven by a software program seeking all the possible cases of automorphism.

$d_{n-(i+1)+j}$. Since these two atoms are contiguous, their sum has the index of the lowest index of the sum elements, then it is $D_{n-(i+1)+j}$ and the condition of automorphism of the sum is met. In a dual manner, if we consider two contiguous maximum elements $D_i \,.\, D_{i+1} = d_{i+1}$, by applying the symmetry we obtain $S_j(D_i \,.\, D_{i+1}) = S_j\, d_{i+1} = d_{n-(i+1)+j}$. But $S_j\, D_i = D_{n-i+j-1}$ and $S_j\, D_{i+1} = D_{n-(i+1)+j-1}$. Since these two maximum elements are contiguous, their product will have the index that is greater from the multipliers, then it is $d_{n-i+j-1} = d_{n-(i+1)+j}$ and the condition of automorphism of the product is met. □

The following theorem analyzes the product of two symmetries.

**Theorem 8** *Two* symmetries $S_j, S_k$ *in* ***2Dn*** *comply with the equation of the product–the successive application of each–* $S_j S_k = R_{j-k}$, *where* $R$ *is the rotation of the lattice.*

**Proof**. If we consider an atom, $S_k\, d_i = d_{n-i+k}$ is obtained. By applying the other symmetry $S_j\, d_{n-i+k} = d_{n-(n-i+k)+j} = d_{i+j-k}$, that is, $R_{j-k}\, d_i$ as we sought to demonstrate. If we consider a maximum element, we obtain $S_k\, D_i = D_{n-i+k-1}$. Then $S_j\, D_{n-i+k-1} = D_{n-(n-i+k-1)+j-1} = D_{i-k+j} = R_{j-k}\, D_i$ as we needed to demonstrate. □

As a consequence of this theorem, it turns out that $S_j S_j = R_0 = I$, identity. As their name suggests, the symmetries are involutory. The following theorem presents the product of a symmetry and a rotation.

**Theorem 9** *If we consider a rotation* $R_j$ *and a symmetry* $S_k$ *in* ***2Dn****, these comply with the equations of the product–their successive application–* $S_k R_j = S_{k-j}$ *and* $R_j S_k = S_{j+k}$.

**Proof**. If we consider an atom, $R_j\, d_i = d_{i+j}$. By applying the symmetry, we obtain $S_k\, d_{i+j} = d_{n-(i+j)+k} = d_{n-i+(k-j)} = S_{k-j}\, d_i$. If we consider a maximum, $R_j\, D_i = D_{i+j}$, applying the symmetry it results in $S_k\, D_{i+j} = D_{n-(i+j)+k-1} = D_{n-i+(k-j)-1} = S_{k-j}\, D_i$, then the first equation is proven. Inversely, $S_k\, d_i = d_{n-i+k}$. Applying the rotation, it results in $R_j\, d_{n-i+k} = d_{n-i+k+j} = S_{j+k}\, d_i$. If we consider a maximum, $S_k\, D_i = D_{n-i+k-1}$. Applying the rotation, it results in $R_j\, D_{n-i+k-1} = D_{n-i+j+k-1} = S_{j+k}\, D_i$, then the second equation is proven. □

This theorem shows that $R_j S_k R_j = R_j S_{k-j} = S_{k-j+j} = S_k$ for any rotation in the lattice. Also, $S_j = R_j S_0$ is obtained, allowing to obtain all of the symmetries in the lattice.[88]

The following theorem shows that there are $2n$ automorphisms in the **3Dn** lattice:[89] there are $n$ rotations and an equal number of symmetries, as in the previous case.

> **Theorem 10** *The* symmetries $S_j$ *in* ***3Dn*** *fulfill the equations* $S_j\, d_i = d_{n-i+j}$, $S_j\, C_i = C_{n-i+j-1}$ *and* $S_j\, D_i = D_{n-i+j-2}$, *the operations are* $n$*-module.*

**Proof.**The demonstrations for the sums of the atoms or the product of contiguous bounds–where the result is a central element–is the same as in the case of Theorem 7 [90], only the case of the sum and the product of central elements must be analyzed. If we consider two contiguous central elements $C_i \,.\, C_{i+1} = d_i$, by applying the symmetry, we obtain $S_j(C_i \,.\, C_{i+1}) = S_j\, d_{i+1} = d_{n-i-1+j}$. But $S_j C_i = C_{n-i+j-1}$ y $S_j C_{i+1} = C_{n-(i+1)+j-1}$, then $S_j C_i \,.\, S_j C_{i+1} = C_{n-i+j-1} \,.\, C_{n-(i+1)+j-1} = d_{n-i+j-1}$, then it is proven. The sum of two contiguous central elements is $C_i + C_{i+1} = D_i$, applying the symmetry we obtain $S_j(C_i + C_{i+1}) = S_j D_i = D_{n-i+j-2}$. But $S_j C_i = C_{n-i+j-1}$ y $S_j C_{i+1} = C_{n-(i+1)+j-1}$, then $S_j C_i + S_j C_{i+1} = C_{n-i+j-1} + C_{n-(i+1)+j-1} = D_{n-(i+1)+j-1} = D_{n-i+j-2} = S_j D_i$ with which the theorem is proven. □

[88] These results are of interest to the study of the group of automorphisms, a subject that the present work does not cover in detail.

[89] This result was directly proven by a software program seeking all the possible cases of automorphism.

[90] The fact that the equations have a constant value in the case of maximums and central elements changes nothing with regards to the demonstration.

The following theorem analyzes the product of two symmetries.

**Theorem 11** *Two* symmetries $S_j$, $S_k$ *in* ***3Dn*** *fulfill the equation of the product–the successive application of each one–* $S_j S_k = R_{j-k}$*, where* $R$ *is the rotation of the lattice.*

**Proof**. The demonstrations for the atoms and central elements match that of Theorem 8 for atoms and maximums, since the equations are the same. The demonstration in the case of maximums is sufficient. If we consider a maximum, we obtain $S_k\, D_i = D_{n-i+k-2}$. Then $S_j\, D_{n-i+k-2} = D_{n-(n-i+k-2)+j-2} = D_{i-k+j} = R_{j-k}\, D_i$ as needed to be proven. □

As a consequence of this theorem, we obtain $S_j S_j = R_0 = I$, the identity. As their name suggests, the symmetries are involutory. The following theorem presents the product of a symmetry and a rotation.

**Theorem 12** *If we consider a rotation* $R_j$ *and a symmetry* $S_k$ *in* ***3Dn****, these fulfill the equations of the product–their successive application–* $S_k R_j = S_{k-j}$ *and* $R_j S_k = S_{j+k}$*.*

**Proof**. The demonstration is similar to that of the previous cases and is omitted for the sake of brevity. □

The previous theorems on automorphisms in **2Dn** and **3Dn** have similar demonstrations. This allows us to state that the theorems are valid for **rDn** –with $r > 1$– in general.

## Cones and intervals

Introducing some new notions is necessary to continue to analyze dialectic lattices.

**Definition 14** *Two elements $a, b$ in a lattice* **L** *allow to define the following sets of elements:*

1. Cone*: a set of $x$ elements that fulfill $x \geq a$;*
2. Inverted cone*: a set of $y$ elements that fulfill $y \leq b$;*
3. Interval*: a set of $z$ elements that fulfill $a \leq z \leq b$.*

These definitions are similar to the definitions of *ideal* and *dual ideal* in lattice theory.[91] Figure 10 presents some examples of the three types of elements we have defined. The cone $x \geq a$, inverted cones $y \leq B$ and $y \leq C$ and intervals $a \leq x \leq B$, $a \leq x \leq C$ and their intersection, which is also an interval, are depicted.

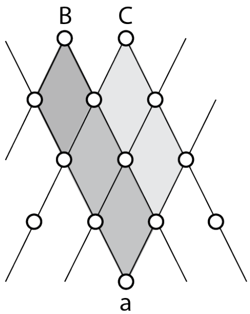


Figure 10: Cones, inverted cones and intervals of the lattice.

As an example, a cone in **Dn** is the set made up of **S** $= (b, 1)$. Similarly, in **2Dn**, Figure 7, **S** $= (b, B, 1)$ is also a cone and in **3Dn**, Figure 9, **S** $= (a, p, A, 1)$ is also one.

The following theorem is fulfilled with regards to these elements.

**Theorem 13** *In a lattice* **L***, the cones, inverted cones and intervals are sub-lattices of* **L***.*

[91] The definition of ideal is: An *ideal* of a lattice **L** is a subset of **S** elements such that if $x, y \in$ **S**, then $x + y \in$ **S**; if $z \leq x$ then $z \in$ **S**. The dual definition is: A *dual ideal* of a lattice **L** s a subset of **S** elements such that if $x, y \in$ **S**, then $x \,.\, y \in$ **S**; if $z \geq x$ then $z \in$ **S**.

**Proof**. Let us consider the case of a cone of vertex $a$. If $x, y$ are two elements in the cone, then $x \geq a$ and $\geq a$, then $x + y \geq a$ y también $x \,.\, y \geq a$ due to monotony properties. In a dual manner, this also applies to the inverted cone, and as a consequence of both results, it is also valid for the interval. $\square$

# Negation

## Monotonic and inverse monotonic functions

The notions of functions with monotony or inverse monotony are basic ideas in the study of negations.

**Definition 15** *A function $f(x)$ defined within a lattice is referred to as* monotonic *if for $x \leq y$, belonging to the lattice, then $f(x) \leq f(y)$; it is referred to as* inverse monotonic $f(x) \geq f(y)$.

It is also usual to say that the function *preserves* or *inverts* the order of the lattice, as expressions equivalent to monotony. The concept of *automorphism* is closely related to the notion of monotony: automorphisms preserve order within the lattice. Due to this close relationship, we can prove the following theorem.

**Theorem 14** *If a transformation of a lattice **L** into **L**, has an inverse and is monotonic, then it is an automorphism.*

**Proof.** If $A$ is the transformation and $A^{-1}$ its inverse, both preserving their order. Since for every pair of elements of the lattice $x + y \geq x$ monotony dictates that $A(x + y) \geq A\,x$. In a similar manner, $A(x + y) \geq A\,y$ is obtained, and due to the monotony of the sum $A(x + y) \geq A\,x + A\,y$ is obtained. If we apply this equation to $A^{-1}$ on the values $A\,x, A\,y$ then $A^{-1}\,(A\,x + A\,y) \geq A^{-1}\,A\,x + A^{-1}\,A\,y = x + y$ and applying $A$ to this equation, then $A\,x + A\,y \geq A(x + y)$ and from this, as the final result, we obtain $A(x + y) = A\,x + A\,y$. In a dual manner the equation of the product is demonstrated and the theorem is proven. □

**Theorem 15** *The inverse of a monotonic (inverse) function $f(x)$ is an (inverse) monotonic function.*

**Proof**. For $x \leq y$, if $f^{-1}(x)$ y $f^{-1}(y)$ are not comparable, $x$ and $y$ would not be either. Let us consider the case of a monotonic function, in the inverse case the demonstration is the same. If it should occur that $f^{-1}(x) \geq f^{-1}(y)$, then applying the monotonic function $f$ we obtain $x \geq y$ against the hypothesis. □

## Intuitive notions on negation

The presentation of natural dialectics and what we know of binary logic rests upon the notion of *negation*. According to the general analysis we have introduced, this notion must be defined within a lattice. We will then investigate, what constitutes a negation?

In technical logics, the notion of negation is usually omitted since most of the times there is no use for it. In the attempts of multi-valued logics at generalization, negation is usually explicitly defined–with no reference to a formal property–by means of an equation which appears arbitrarily. It is usual, however, for these definitions to meet the De Morgan property despite the latter not being thought to represent an essential aspect of negation.

At first sight, it may appear that a negation must be defined by its *meaning*, but this is not the case. A mix-up of concepts which it is not fall prey to to engage in is to blame for this. Negation is a logical operation and should only be defined by its *formal properties*. There are four dialectic concepts which are related, but different. First, there is the concept of *negation*. Second, the concept of *logical opposites*. Third, there is the notion of *material opposites*. Finally, to complete the scenario, there is the idea of *penetration of opposites* or *unity and struggle of opposites*.

In the imprecise formulations of dialectics, these differing ideas are usually mistaken for one another. A first step towards precisely defining logical content consists in separating them. In this section we will take care of the meaning of the first two. Further ahead we will cover the concepts of material opposites and penetration of opposites.

## The formal properties of negation

The notion of negation extends an idea developed by Boolean logic. Negation is a *unary* operation, defined over all the elements of the lattice. If $x$ is an element of lattice **L**, the notation $N\,x$ will be used to designate a negation of $x$. We will use this notation since there *is more than one negation* in a dialectic lattice and therefore, it is not convenient to use the classic symbol $\neg\, x$. The different negations are written as $N_1, N_2, \cdots$.[92]

If we take a logical value and successively apply a negation, a series of logical values are obtained which, at some point, must come back onto themselves and lead to the *starting logical value*. This is a generalization of the Hegelian negation. This demand translates the property of the double negation which in Boolean logic matches the statement, and of the triple negation, which in Hegelian logic somehow leads to the starting point. For this reason, negations are unary operations *with an inverse*. The inverse function of $N$ will be indicated as $N^{-1}$.

The requirement for a negation to have an inverse characterizes it very poorly from an algebraic standpoint. Formally speaking, there is another fundamental logical property: *the De Morgan property*. This property exists in the universe of statements, before applying the R-homomorphism. In human thought, this property is used spontaneously.[93] The negation is fully characterized through the De Morgan property.

[92] The multiplicity of negations is not uncommon in the natural sciences. Let us consider the notion of "opposite". It is clear that, given any element or action, there are a number of possible opposites. Therefore, for example, what is the opposite of life? A list can be drafted immediately: death, a comatose state, a tormented soul, a doomed soul, a soul in paradise, reincarnation. There will be as many opposites as there are beliefs on life. The opposite of love calls for a long list as well: hate, religious ascetic, insane, dead and many other possibilities.

[93] As an example, it is interesting to note that the De Morgan property exists in Spanish as a natural and extraordinarily precise occurrence. In fact, the negation of the sentence "o A o B" [either A or B] is the sentence"ni A ni B" [neither A nor B] which expresses the De Morgan property, if we believe that"ni" is a contraction of "no y". [not and]. In other languages this relationship is not as perfect.

**Definition 16** *A negation $N$ in a lattice is a unary operation, with an inverse, which meets the De Morgan property: $N(x + y) = Nx \,.\, Ny$ and also $N(x \,.\, y) = Nx + Ny$.*

The De Morgan property defines an anti-isomorphism within the lattice. It indicates that there is a certain "symmetry" within the structure of the logical values. It is also linked to a property of preservation of the order defined in the lattice.

**Theorem 16** *Every negation $N$ defined in a lattice is an* inverse monotonic *function (or one that inverts the order).*

**Proof**. If a function meets the De Morgan property for two logical values that verify that $x \leq y$, then applying the elementary properties we have $x + y = y$ and also $x \,.\, y = x$. Applying the De Morgan property to the previous expressions, we have $N\,x \,.\, N\,y = N\,y$ and also $N\,x + N\,y = N\,x$ and from any of these two expressions it follows immediately that $N\,y \leq N\,x$, as needed to be proven. □

**Theorem 17** *If a function $f(x)$ has an inverse and inverts the order in a lattice, then $N\,x = f(x)$ is a negation.*

**Proof**. Let us consider two elements in the lattice. Since we have that $x + y \geq x$, due to the inverse monotony property this leads to $f(x + y) \leq f(x)$ and from $x + y \geq y$ we have that $f(x + y) \leq f(y)$. From the monotony of the product we have that $f(x + y) \leq f(x) \,.\, f(y)$. In a dual manner we can prove that $f(x \,.\, y) \geq f(x) + f(y)$. Let us consider now $f^{-1}$, inverse of $f$, which is also an inverse monotonic function, and apply this property to $f(x)$ and $f(y)$, then $f^{-1}(f(x) \,.\, f(y)) \geq f^{-1}(f(x)) + f^{-1}(f(y)) = x + y$. If we apply $f$ to the formula, keeping inverse monotony in mind, we have that $f(x) \,.\, f(y) \leq f(x{+}y)$. Combining the results, we obtain $f(x) \,.\, f(y) = f(x{+}y)$, one of the De Morgan equations. In a dual manner we obtain the other property and it is proven that $f$ is a negation. □

**Theorem 18** *Every negation complies with:* $N\,0 = 1$ *and* $N\,1 = 0$.

**Proof.** If $x$ is a lattice element and we define $z = N^{-1}\,x$, then we have $0\,.\,z = 0$, then, for every $x$, we have $N\,0 + x = N\,0$. Replacing $x = 1$ it yields that $N\,0 = 1 + N\,0 = 1$. The other equation is demonstrated in a dual manner. □

This result allows us to take another step towards interpreting logical values in a lattice. We can assimilate the uppermost value of the lattice, 1, to the logical value "true", and the lowermost value, 0, to the logical value "false", just as in the classic binary interpretation. With this presentation, the sub-lattice made up of 0 and 1, with any negation, is indistinguishable from binary logic. Through this argument, we begin to interpret the meaning of the logical values of the lattice. With regards to the values "true" and "false", the defined negations behave as expected.

Negations include some special cases, which are useful in logic and are referred to as *strict negations*.

**Definition 17** *A negation* $N$ *defined in lattice* **L** *is referred to as a* strict negation *if it turns every element into a strict opposite, that is, if for any* $x$ *we have:* $x + N\,x = 1$ *and* $x\,.\,N\,x = 0$.

The Hegelian negation $(01)(t\,a\,s)$ is strict. The following theorem is verified.

**Theorem 19** *The composition of negations has the following properties:*

1. *The product of an even number of negations is an automorphism in **L**; the product of an odd number of negations is a negation.*
2. *Every negation of **L** can be obtained as the product of any fixed negation $N_0$ for every automorphism of **L**.*
3. *If $N_1$ and $N_2$ are two negations in **L** then $N_3 = N_1^{-1}\, N_2\, N_1$ is a negation. If $N_2$ is strict, then $N_3$ also is.*
4. *If $N$ is a negation and $A$ is an automorphism, then $A^{-1}\, N\, A$ is also a negation. If the negation is strict, $A^{-1}\, N\, A$ also is.*
5. *If $N$ is a strict negation, $N^{-1}$ also is.*

**Proof.** We will demonstrate this one item at a time.

1. This statement is immediate due to the monotony properties.
2. If we consider a specific negation $N_0$ and any negation $N$, it is clear that $N = (N\, N_0^{-1})\, N_0$ where $A = N\, N_0^{-1}$ is an automorphism, as needed to be proven.
3. Due to property 1, $N_3$ is a negation. This can also be proven directly. Let us consider

$$
\begin{aligned}
N_3\,(x+y) &= N_1^{-1}\, N_2\, N_1(x+y) = N_1^{-1}\, N_2\,(N_1\, x \,.\, N_1\, y) = \\
&= N_1^{-1}\,(N_2\, N_1\, x + N_2\, N_1\, y) = \\
&= N_1^{-1}\, N_2\, N_1\, x \,.\, N_1^{-1}\, N_2\, N_1\, y = N_3\, x \,.\, N_3\, y
\end{aligned}
$$

   In an equal manner it is proven that $N_3(x\,.\,y) = N_3\, x + N_3\, y$. If $N_2$ is a strict negation, for every $x$ we have that $x + N_2\, x = 1$. Multiplying to the left by $N_1^{-1}$ and to the right by $N_1$ we have that $x + N_1^{-1}\, N\, N_1\, x = 1$. Applying this reasoning to the dual case, it is proven.

4. $A^{-1}N\,A$ is a negation given that $A^{-1}N\,A(x+y) = A^{-1}N(A\,x+A\,y) = A^{-1}(N\,A\,x\,.\,N\,A\,y) = A^{-1}N\,A\,x\,.\,A^{-1}N\,A\,y$. In a similar manner it is proven for $x\,.\,y$. It is clear that $A\,x + N\,A\,x = 1$ if $N$ is a strict negation. Applying $A^{-1}$ it yields that $x + A^{-1}\,N\,A\,x = 1$. This is proven in a similar manner for $x\,.\,y$.

5. $N^{-1}$ is a strict negation given that $x + N\,x = 1$, then, applying the negation $N^{-1}$ it yields that $N^{-1}x\,.\,x = 0$ and in a similar manner the dual case $N^{-1}x + x = 1$ is proven.

This proves all the cases. □

**Definition 18** *The degree of a negation $N$ is referred to as the smallest number of times it is necessary to apply $N$ in order to obtain an identical transformation. The degree is an even number.*

For example, in Hegelian lattice **D3** negation (0 1)(t a s) has degree 6, but negation (0 1)(t a) has degree 2. Not all the negations in a lattice have the same degree.

**Theorem 20** *In a lattice **rDn** all the negations of an element of logical value $s$ are elements of logical value $r - s + 1$.*

**Proof**. Let us consider an element $d_{s,t}$ from the lattice. Given the chain:

$$0 < d_{1,p} < d_{2,q} < \cdots < d_{s,t} < d_{s+1,u} < \cdots d_{r,z} < 1.$$

This chain has $s - 1$ contiguous elements between 0 and element $d_{s,t}$ and $r - s$ elements until 1, in total there are $r$ dialectic elements. Applying a negation $N$ we have:

$$1 > N\,d_{1,p} > N\,d_{2,q} > \cdots > N\,d_{s,t} > N\,d_{s+1,u} > \cdots > N\,d_{r,z} > 0.$$

Then, $N\,d_{s,t}$ has $r - s$ elements until 0, therefore, its logical level is $r - s + 1$ as needed to be proven. □

As a corollary of this theorem we have that if $r = 2s - 1$ then the negation of a central element of logical value $s$, is of logical value $s$.

## Dialectic opposites

It is necessary to distinguish the notion of *opposites* from that of *strict opposites*, just as we have made a distinction between *negation* and *strict negation*.

> **Definition 19** *The element $y$ of a dialectic lattice is referred to as a* simple opposite *or simply an* opposite*, if there is $x$ and a negation $N$ such that $y = N\,x$.*

The following theorem establishes some properties of opposites.

> **Theorem 21** *If the element $y$ of a dialectic lattice is an opposite of $x$, then::*
>
> 1. *the element $x$ is an opposite of $y$;*
> 2. *the element $Nx$ is an opposite of $Ny$, where $N$ is any negation;*
> 3. *the element $Ny$ is an opposite of $Nx$.*

**Proof**. In property 1, by the definition of opposites, there is a negation $N_i$ such that $y = N_i\,x$, then it occurs that $y = N_i^{-1}\,x$ and they are opposites. In property 2, applying $N$ to the already known equation, we have that $N\,y = N\,N_i\,x$, then $N\,y = (N\,N_i\,N^{-1})N\,x$, but due to Theorem 19 $N\,N_i\,N^{-1}$ is a negation, then it is proven. Property 3 is a consequence of 1 and 2. □

## Examples in **D3**

In order to consolidate the ideas introduced, let us consider Hegelian lattice **D3** from Figure 2 and the negation defined as $N\,0 = 1$, $N\,1 = 0$, $N\,t = a$, $N\,a = s$, $N\,s = t$, where $t, a, s$ are, respectively, *thesis*, *antithesis* and *synthesis*. Using the replacement notation, this negation can be written as:

$$N = (0\,1)(t\,a\,s).$$

Since a negation in **L** is a permutation of its elements, a notation similar to the one used in replacement *groups* can be employed. In this way, it is indicated that 0 becomes 1 and reciprocally, $t$ becomes $a$, becomes $s$ and $s$ becomes $t$. Each list enclosed in parentheses indicates a closed cycle. If any element does not appear, it means that the operation transforms it into itself.

In the lattice considered, 6 negations can be defined which correspond to the 6 possible permutations of the elements t, a, s. These negations are:

$$(0\,1)\;\;(0\,1)(t\,a)\;\;(0\,1)(t\,s)\;\;(0\,1)(a\,s)\;\;(0\,1)(t\,a\,s)\;\;(0\,1)(t\,s\,a).$$

The last two negations are strict. The automorphisms are also 6 and they are:

$$I\;\;(t\,a)\;\;(t\,s)\;\;(a\,s)\;\;(t\,a\,s)\;\;(t\,s\,a)$$

where I is the identical transformation. The set of the 12 transformations make up an *algebraic group*[94] that we refer to as $\mathbf{G}_L$, the group of transformations of the lattice **L**.

As we will see, within the same lattice **L** there may be negations in a broad sense as well as strict negations. It is important to make this distinction when studying some problems. In the exposition that follows it will be explicitly indicated whether a negation is strict in the context where it is used.

Only those negations which exchange the three elements within the Hegelian lattice **D3** are strict negations, and are inverse with regards to one another. Not every lattice has strict negations. Therefore, for instance, there are none in **3D5**, but they do exist in **3D4**.[95]

[94] A group is a set of elements **G**, such that if $x, y \in$ **G**, then an operation *product* $x\,y$ is defined–associative, although not necessarily commutative–and there exists an element $I \in$ **G** with the property $I\,x = x\,I = x$. Also, every element $x$ has an inverse element $x^{-1}$ such that $x\,x^{-1} = x^{-1}\,x = I$.

[95] In **3D4** the strict negation is $N = (0\,1)(a\,B)(b\,D)(c\,D)(d\,A)(p\,r)(q\,s)$ which is also involutory or degree 2.

It is interesting to note that there are negations–as occurs with the first four in Hegelian lattice **D3**– that have *elements that match their negation*.

This situation is not new in logic, since modal logics [58] already possessed central elements. It is also not new to dialectics, and thus occurs in the classical statements by Heraclitus, such as:

> *The way up and the way down are one and the same.*[45, Diels #108]
>
> *For the wool–carder the straight and the winding way are one and the same.* [45, Diels #111]
>
> *It is one and the same thing to be living and dead, awake or asleep, young or old. The former aspect in each case becomes the latter, and the latter becomes the former, by sudden unexpected reversal.* [45, Diels #113]

All of the cases express a coincidence between an idea and the negation of this idea. This point of view of Heraclitus's dialectics poses no difficulties for the logic we are studying–even if we went to an extreme and took the coincidence in a strict and literal sense, there are still elements and negations for which it is verified.

A logic is defined whenever a lattice **L** and a negation $N$ are specified. In this section, we will deal with a collection of lattices and negations whose informal characterization is that they have *logical interest*.

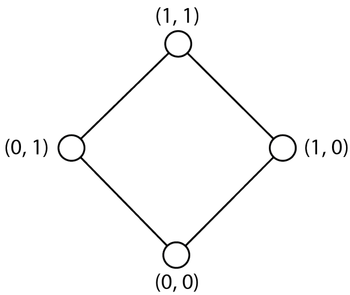


Figure 11: *Yin-yang* dialectics as the product of Boolean logics.

The *yin-yang* is the producto –see Definitionn 9, page 84– B$\times$B of two simple Boolean lattices, see Figure 11.

The *distributive* property in a lattice has fundamental logical consequences.

**Definition 20** *A lattice is* distributive *if, for any three elements* $x, y, z$*, the property* $x\,.\,(y+z) = x\,.\,y + x\,.\,z$ *is met.*

With this definition, the following theorem is valid.

**Theorem 22** *If a lattice* **L** *is distributive, then there is a single strict negation* $N$ *with the involutory property* $N N\,x = x$.

**Proof**. Let $N_1$ and $N_2$ be two strict negations. It is clear that $N_1\,a = N_1\,a\,.\,1 = N_1\,a\,.\,(a + N_2\,a) = N_1\,a\,.\,a + N_1\,a\,.\,N_2\,a = N_1\,a\,.\,N_2\,a$. It then follows that $N_1\,a \leq N_2\,a$. In a symmetrical manner, we have that $N_2\,a \leq N_1\,a$, then $N_2\,a = N_1\,a$ for every $a$. Due to Theorem 19, $N^{-1}$ is a strict negation. Then $N$ y $N^{-1}$ match and $N^{-1}a = N\,a$ resulting in $N^2 a = a$. □

It is worth noting that *the reciprocal is not true*. As we will see further ahead, there are strict involutory negations in non-distributive lattices. It also occurs that in a distributive lattice such as $\mathbf{B}^2$ there is a negation that is not strict, such as the negation $N = (0\,1)$.

From a physical standpoint, this establishes that the logic of the spin, by being non-distributive, cannot be assimilated to a Boolean logic. This makes for the fundamental "illogic' of the mechanics of elementary particles.

Another major conclusion that we will not prove but can be found in [16], establishes that any Boolean logic of a finite number of elements is a power $\mathbf{B}^n$ of the simple Boolean logic.

## Unit functions in dialectic lattices

The study of logical functions is the next step in the construction of a dialectic. Before analyzing the functions that are of specific interest to the subject, it is convenient to make a more general analysis of logical functions. This analysis begins with a major observation. The functions that can be built in a lattice by means of constant values, variables and the two operations, are monotonic functions because all of these operations are as well. In order to construct functions which are not monotonic, it is necessary to incorporate negations, which are inverse monotonic functions. This fact stresses the importance of the negation function.

In order to analyze the structure of logical functions that can be constructed by means of the two operations and a negation within a lattice, we will begin by defining *unit functions*.

**Definition 21** *A* unit function $U(x, a)$ *of a lattice* **L** *and an element* $a$ *is a function such that for* $x \in$ **L** *is verified that for* $x = a$*,* $U(x, a) = 1$*, and for* $x \neq a$*,* $U(x, a) = 0$*.*

Some important theorems for the lattices studied can be proven through this definition.

**Theorem 23** *In every dialectic lattice of rank 1 and degree* $n > 2$*, the functions* $U(x, 1)$ *and* $U(x, 0) = U(N\,x, 1)$ *can be built. For every element* $p$ *in the lattice,* $p \neq$ *0, 1, the following is valid:* $U(x, p) = N\,U(p\,.\,x, 0)\,.\,N\,U(p + x, 1)$*, where* $N$ *is a negation.*

**Proof**. The lattice has, at least, three atoms $a, b, c$, opposite among themselves, since $n \geq 3$. The following function can be constructed:

$$U(x, 1) = (a\,.\,x + b)\,.\,(a\,.\,x + c)\,.\,(b\,.\,x + a)\,.\,(b\,.\,x + c).$$

For $x = 0$ the function has the value $U(0, 1) = b\,.\,c\,.\,a\,.\,c = 0$. For $x \neq a, b$ then $U(x, 1) = b\,.\,c\,.\,a\,.\,c = 0$. For $x = a$ then $U(x, 1) = (a + b)\,.\,(a + c)\,.\,a\,.\,c = 0$. The same as with $a$ occurs for $x = b$ given that the function is symmetrical in these parameters. Finally, for $x = 1$, $U(x, 1) = (a + b)\,.\,(a + c)\,.\,(b + a)\,.\,(b + c)$ =1. It is clear that the function $U(x, 0) = U(N\,x, 1)$ is worth 1 only when $N\,x = 1$, then only when $x = 0$. The function $U(p\,.\,x, 0) + U(p + x, 1)$ for $x = p$ is worth 0 and for any dialectic value different from p it is worth 1 given that $p \, . \, x = 0$ y $p + x = 1$. For $x = 0$ it is worth 1 due to the first summand and for x = 1 it also has value 1 due to the second summand. Then, the negation of the summand, due to De Morgan, proves the result. □

> **Theorem 24** *In every dialectic lattice* ***rDn*** *wirh rank* $2 \leq r < n - 1$ *the unit functions* $U(x, 1)$ *y* $U(x, 0) = U(N\,x, 1)$ *can be constructed.*

**Proof**. Given $M$ a maximum of the lattice. Due to the condition of rank, this maximum has at least two logical opposite atoms. In fact, we will consider the $r$ atoms which are smaller than $M$. Due to the property of $n$, there are at least two atoms outside of this set and due to how they were obtained, they are logical opposites. If $a$ and $b$ are these logical opposite atoms of $M$, then, the following function can be constructed:

$$f(x, M) = (M \, . \, x + a) \, . \, (a \, . \, x + M) \, . \, b.$$

For $x = 1$, it is worth $(M+a) \, . \, (a+M) \, . \, b = b$. If $x \leq M$, the function is worth $(x + a) \, . \, M \, . \, b = 0$. If $x$ is not comparable ton $M$, then the function is worth $a \, . \, (a \, . \, x + M) \, . \, b = 0$. Let us consider now all the maximum elements $M_i$ in the lattice and add up all the functions $f_i$, then:

$$U(x, 1) = f_1(x, M_1) + \cdots + f_n(x, M_n)$$

since for every $x \neq 1$ it is worth 0 since all the summands are worth 0. For $x = 1$ it is worth 1 because it is the sum of all the atoms in the lattice, as needed to be proven. It is clear that the function $U(x, 0) = U(N\,x, 1)$ is worth 1 only when $N\,x = 1$, lthen only when $x = 0$. □

> **Theorem 25** *In every dialectic lattice* ***rDn*** *with rank that meets* $2 \leq r < n - 1$ *the unit functions* $U(x, p)$*, can be constructed, where* $p$ *is any element in the lattice.*

**Proof**. The theorem has already been proven for 0 and 1. Be $a_i$ and $M_j$ respectively, the $s$ atoms and $t$ maximum elements that meet

$a_i \leq p$ y $M_j \geq p$. Given the function:

$$g(x) = U(a_1 \,.\, x, 0) + \cdots + U(a_s \,.\, x, 0).$$

The function $g(x)$ –which can only take on value 0 or 1– is 0 if all the summands are 0, for which it must occur that all the products $a_i \,.\, x$ must be different from 0. Given that the elements $a_i$ are atoms, it must occur that $a_i \,.\, x = a_i$ that is, $x \geq a_i$ that is, $x \geq a_1 + \cdots + a_s = p$. Reciprocally, if $x \geq p$ the function is worth 0 since all the products with atoms are different from 0. Contrarily, for every other value of $x$, the function is worth 1. Now, given the function:

$$h(x) = U(M_1 + x, 1) + \cdots + U(M_t + x, 1).$$

The function $h(x)$ –which can only take on value 0 or 1– is 0 if all the summands are 0, for which it must occur that all the sums $M_j + x$ must be different from 1. Given that the elements $M_j$ are maximum elements, it must occur that $M_j + x = M_j$ where it follows that $x \leq M_j$ that is, $x \leq M_1 \,.\, \cdots \,.\, M_t = p$. Reciprocally, if $x \leq p$ the function is worth 0 since all the summands with maximum elements are different from 1. If we now have the function:

$$g(x) + h(x)$$

This function–which can only take on value 0 or 1–is 0 when it holds true that $a_1 + \cdots + a_s = p \leq x \leq p = M_1 \,.\, \cdots \,.\, M_t$. Then, the only value that meets these inequalities is $x = p$. Then, the function $U(x, p) = N\ (g(x) + h(x))$ is worth 1 only when x = p, as needed to be proven. □

One of the consequences of the existence of unit functions is the possibility of building functions that take on a desired set of values, as shown in the following theorem.

**Theorem 26** *If a lattice **L** has, for every element $a$, a unit functiona $U(x, a)$, then any function can be constructed on this lattice, through the unit functions and logical operations, a function $f(x)$ such that for each pair of values $a_i, b_i \in$ **L** then $f(a_i) = b_i$.*

**Proof.** If we have a table that equates each of the values $a_i \in$ **L** to the valuer $b_i \in$ **L** -including the values 0 and 1 1–, there is a function $f(x)$ such that $f(a_i) = b_i$ given by:

$$f(x) = b_1 \,.\, U(x, a_1) + \cdots + b_s \,.\, U(x, a_s)$$

which takes on the indicated value, $b_i$ for each $a_i$. □

**Theorem 27** *Given **H**, a homomorphism that transforms a lattice **L** in **L'** and given $N$ a negation defined in **L**. For every element $x' \in$ **L'** that is an image of the element $x \in$ **L**, with **H**:$x \rightarrow x'$, a negation $N'$ can be defined in **L'** as $N'x'$ =**H**:$Nx$.*

**Proof.** It is clear that every element of **L'**, since it is an image of an element of **L** has a negated element defined. It is only necessary to prove the De Morgan property for $N'$. Let us consider $x', y' \in$ **L**, it is clear by the definition of homomorphism that **H**:$(x + y) \rightarrow x' + y'$. Then, $N'(x' + y')$= **H**:$(N(x + y))$ = **H**:$(Nx \,.\, Ny)$ = **H**:$Nx$ . **H**:$Ny$ = $= N'x' \,.\, N'y'$. In a dual manner, the dual case is proven. □

**Theorem 28** *If a lattice **L** complies with the conditions of Theorems 23 and 24 then it does not have* R-homomorphisms *except a trivial one.*

**Proof.** Given **H** an R-homomorphism. There must be at least two elements $a, b \in$ **L** such that **H**:$a$ = **H**:$b$ for the image of **L** to have less elements than **L**. Due to Theorem 26, a logical function $f(x)$ can be constructed, by means of the $+ \,.\, N$ that takes on the values $f(a) = 1$ y $f(b) = 0$. The **H** homomorphism allows us to define the function $f'(x')$ =**H**:$f(x)$ by the application of the expression $f(x)$ constructed by the operations $+ \,.\, N$, as per Theorema 27. We will then have that 1'= **H**:$f(a)$ = $f'$(**H**:$a$) = $f'$(**H**:$b$) = **H**:$f(b)$ = 0', then the homomorphism is trivial. □

This theorem shows that the lattices that comply with Theorems 23 and 24 do not have homomorphisms that maintain the logical properties but have less elements. They are lattices that construct a logic that cannot be further "simplified", the ultimate image of the universal homomorphism that constructs the logic.

It is convenient to define other unit functions that are useful in constructing logical functions. To that end, we will simply define as $U(x)$ the unit function $U(x, 1)$.

**Theorem 29** *The unit function $D(x)$ with value 1 can only be constructed if $x$ has a dialectic value.*

**Proof**. It is clear that $D(x) = N\,U(x)\,.\,N\,U(Nx)$ where $N$ is a negation, given that $N\,U(x) = 1$ for every $x \neq 1$ and $N\,U(Nx) = 1$ for every $x \neq 0$.Then, the product is worth 1 if and only if it is a dialectic value. □

For two-variable functions, the specific unit functions that are worth 1 in each of the functional regions indicated can also be constructed. Table 5 shows the different situations where, for instance, $U_{1d} = 1$ if $x = 1$ and $y$ have a dialectic value.

Table 5: Simple scheme of unit functions.

| | 0 | dialectic values | 1 |
|---|---|---|---|
| 0 | $U_{00}$ | $U_{0d}$ | $U_{01}$ |
| dialectic values | $U_{d0}$ | $U_{dd}$ | $U_{d1}$ |
| 1 | $U_{10}$ | $U_{1d}$ | $U_{11}$ |

The different functions are defined by the following expressions depending on $U, D, N$:

$$\begin{array}{ll} U_{00} = U(N\,x)\,.\,U(N\,y) & U_{d0} = D(x)\,.\,U(N\,y) \\ U_{0d} = U(N\,x)\,.\,D(y) & U_{dd} = D(x)\,.\,D(y) \\ U_{01} = U(N\,x)\,.\,U(y) & U_{d1} = D(x)\,.\,U(y) \\ U_{10} = U(x)\,.\,U(N\,y) & U_{1d} = U(x)\,.\,D(y) \\ U_{11} = U(x)\,.\,U(y) & \end{array}$$

**Theorem 30** *Every function $f(x, y)$ can be broken down as follows:*

$$f(x, y) = g_{00}\,.\,U_{00} + g_{d0}\,.\,U_{d0} + \cdots + g_{11}\,.\,U_{11}$$

*where $g_{00}, g_{d0}, \cdots, g_{11}$ are functions of the variables $x, y$. the function $f(x, y)$ is invariable in automorphism A, then so are the functions $g_{00}, g_{d0}, \cdots, g_{11}$.*

**Proof**. In fact, $A\,f(x, y) = f(A\,x, A\,y)$, must be met, where $A$ is the automorphism considered. All the unit functions are invariable, then it must be verified that:

$$\begin{aligned} &A\,g_{00}(x, y)\,.\,U_{00} + \cdots + A\,g_{11}(x, y)\,.\,U_{11} = \\ &\qquad = g_{00}(A\,x, A\,y)\,.\,U_{00} + \cdots + g_{11}(A\,x, A\,y)\,.\,U_{11}. \end{aligned}$$

Given that this representation is unique in the region considered, it must occur that $A\,g_{00}(x, y) = g_{00}(A\,x, A\,y), \cdots, A\,g_{11}(x, y) = g_{11}(A\,x, A\,y)$ and all the functions in the breakdown are also invariable in the corresponding area. □

Table 6 presents a simpler notation for the intrinsic functions of two variables in a dialectic lattice.

The structure of Table 5 can be generalized to the lattices of rank higher than 1, as shown in Table 7. It is considered that *dial*$_1$ and *dial*$_2$ are sets of dialectic elements *of equal logical level*. The result is general no matter the number of logical levels in the lattice.

In order to generalize the results it is enough to prove that the unit functions that appear in the table exist. Some match the previous, such as $U_{01}$ and similar, but others are new. If we take the case

Table 6: Truth table of a generic invariant function.

| | 0 | dialectic values | 1 |
|---|---|---|---|
| 0 | 0,1 | $f_1(y)$ | 0,1 |
| dialectic values | $f_4(x)$ | $g(x, y)$ | $f_2(x)$ |
| 1 | 0,1 | $f_3(y)$ | 0,1 |

Table 7: Composite scheme of unit functions.

| | 0 | dial$_1$ | dial$_1$ | 1 |
|---|---|---|---|---|
| 0 | $U_{00}$ | $U_{0d_1}$ | $U_{0d_2}$ | $U_{01}$ |
| dial$_1$ | $U_{d_1 0}$ | $U_{d_1 d_1}$ | $U_{d_1 d_2}$ | $U_{d_1 1}$ |
| dial$_2$ | $U_{d_2 0}$ | $U_{d_2 d_1}$ | $U_{d_2 d_2}$ | $U_{d_2 1}$ |
| 1 | $U_{10}$ | $U_{1d_1}$ | $U_{1d_2}$ | $U_{11}$ |

$U_{d_1 d_2}$, as before, this function is the product of the simple functions $D_{d_1}(x) \, . \, D_{d_2}(y)$. To prove the existence of these functions, for instance in the case of the former, we will consider a dialectic element $a$, of which we know that the unit function $U(x, a)$. We then have that:

$$D_{d_1}(x) = U(x, a) + U(x, A\,a) + \cdots + U(x, A^p a)$$

where $I = A^0, A, \ldots, A^p$ are all the automorphisms in the lattice and, therefore, they generate all the values of equal logical level as *dial*$_1$. The same thing happens for *dial*$_2$.

As in the simple case, the breakdown of a function–as the sum of functions by means of unit functions–is unique. In the case of a function invariant in rotation, each of the composing functions must also be rotationally invariant.

## The group of negations and automorphisms

The following definition establishes the essential quality of the functions defined within a lattice: their invariance in an automorphism.

**Definition 22** *Given $f(x, \ldots, z)$, a function of lattice **L** and $A$ a non-trivial automorphism of the lattice. $f$ is said to be invariant in $A$ if it is verified that $A\, f(x, \ldots, z) = f(A\, x, \ldots, A\, z)$.*

This property is equivalent to the following functional diagram (for the sake of simplicity, it is represented as a single variable):

$$\begin{array}{ccccc} & & F & & \\ & x & \rightarrow & F(x) & \\ A & \downarrow & & \downarrow & A \\ & y & \rightarrow & F(y) & \\ & & F & & \end{array}$$

**Theorem 31** *If function $f(x, \ldots, z)$ in a lattice is invariant in $A$, it is invariant for every automorphism $A^s$.*

**Proof**. It is clear that $A\,(A\, f(x, \ldots, z)\,) = A\, f(A\, x, \ldots, A\, z) = f(A^2\, x, \ldots, A^2\, z)$ and so on for the subsequent applications of $A$. □

An automorphism applied to the arguments of a function yields the same result as if it were applied it to the result of the function. In this way, automorphisms–rotations, for example, in the case of dialectic lattices–establish the formal equivalence between their dialectic elements. This property is essential in the functions used in logic.

**Theorem 32** *Given $f(x, \ldots, z)$ an invariant function in lattice **L**. The set of automorphisms that comply with $A\, f(x, \ldots, z) = f(A\, x, \ldots, A\, z)$ is a subgroup of $\mathbf{G}_L$.*

**Proof**. Given $A$ and $B$ two automorphisms that meet $A\, f(x, \cdots, z) = f(A\, x, \ldots, A\, z)$ y $B\, f(x, \ldots, z) = f(B\, x, \ldots, B\, z)$. Then:

$$B\, A\, f(x, \ldots, z) = B\, f(A\, x, \ldots, A\, z) = f(B\, A\, x, \ldots, B\, A\, z).$$

It is therefore proven that the product of two automorphisms belongs to the set. Let us now consider:

$$A\, f(A^{-1}\, x, \ldots, A^{-1}\, z) = f(A\, A^{-1}\, x, \ldots, A\, A^{-1}\, z) = f(x, \ldots, z)$$

then it is clear that $f(A^{-1}\, x, \ldots, A^{-1}\, z) = A^{-1}\, f(x, \ldots, z)$, which proves that the inverse of an automorphism also belongs to the set. The identity automorphism also belongs to the set. □

It is evident that negations and automorphisms comprise a group $\mathbf{G}_L$ of transformations of the lattice as a consequence of Theorem 19. It follows that a set of very simple properties are obtained from this group.

The product of strict negations is not necessarily a strict negation. There are many examples of this. We will introduce a method for composing lattices which is specific to dialectics.[96]

**Definition 23** *If we consider two lattices (disjointed, with no common elements),* $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$. *A* dialectic composition, *or simply* composition *of two lattices* $\boldsymbol{L}_1$ *and* $\boldsymbol{L}_2$ *refers to the lattice* $\boldsymbol{L}_1 \uplus \boldsymbol{L}_2$ *made up of all the elements in each lattice, with its own relations of order, but sharing the elements* 0 *and* 1, *as shown in Figure 12.*

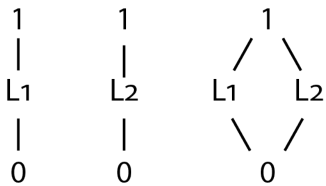


Figure 12: Dialectic composition of lattices $\mathbf{L}_1$ and $\mathbf{L}_2$.

As follows from this, each lattice keeps all of its properties of negation, isomorphism and homomorphism, they only go on to share the external elements. The dialectic composition $\uplus$ can be extended to $n$ lattices and it is a commutative and associative operation.

[96] This method of lattice composition is unusual in mathematics.

It then follows that the group $\mathbf{G}_L$ of the composition of two lattices that have groups $\mathbf{G}_{L_1}$ and $\mathbf{G}_{L_2}$ is the *direct product* of groups $\mathbf{G}_{L_1} \times \mathbf{G}_{L_2}$. In some way, an inverse result exists.

**Theorem 33** *If $N$ is a negation in a lattice **L**, then automorphism $A = N\,N$, which transforms an atom into an atom of the lattice, generates a replacement $S$ between the atoms, that defines a set of sub-dialectic lattices $\mathbf{L}_i$ such that their composition matches $\mathbf{L}$=$\mathbf{L}_1 \uplus \cdots \uplus \mathbf{L}_s$.*

**Proof.** $S$ is a replacement between the atoms, and, as such, is the direct product of several partial replacements between them. Given $S = S_1 \times \cdots \times S_s$ where each replacement $S_i$ has a single cycle. Atoms $a_{j\,i} \in S_i$ generate a lattice $\mathbf{L}_i$ by successive additions between them. The uppermost elements obtained by negation are also a part of it. The negation $N$ is a negation between the elements of this sub-lattice which is strict. Then, $\mathbf{L}_i$ is a dialectic lattice–because it has a strict negation and each element is the sum of its atoms or a product of its maximum elements–which has a group $\mathbf{G}_{P_i}$ which is a direct factor of the group $\mathbf{G}_P$ de **L**. By the successive application of this procedure, the various sub-lattices that make up **L** can be found. □

It is worth noting that the theorem *does not establish* that $N$ –or its derivatives $A$ or $S$– determine the lattice, *it only establishes its nature as a composite.* In fact, lattices **2D5** and **3D5** generate the same replacement $S$ among their five atoms, a rotation of the five elements, and, however, the lattices are not equal.

It is also worth pointing out a special characteristic of rank-1 dialectic lattices. As an example, although one which is, in general, valid, we will consider lattice **D5** and the strict negation $N = (0\,1)(a\,b)(c\,d\,e)$. By applying the previous result, it is shown that the relation **D5** = **D2** $\uplus$ **D3**. is met. This can be generalized in many other ways to (almost) all rank-1 dialectic lattices.

This type of lattice composition may be of interest when applying dialectics to the relationships between many apparently contradictory scientific theories. A commentary on this condition appears further ahead, towards the end of the book.

## Negations in **Dn**

The *alphabetic notation* will (almost) always be used for the elements –$a, b, \cdots$– despite the fact that this suggests an order that the lattice atoms *do not have*. Occasionally, for **D3** we will use the Hegelian notation $t, a, s$.

This case is very special and very simple. Every permutation of the atoms generates a negation, with the simple addition of transforming 0 and 1 between each other. In fact, if $a$ and $b$ are two different atoms, $a \,.\, b = 0$ and $a + b = 1$. Any permutation $N$ transforms $a, b$ into atoms $N\,a$ and $N\,b$, which are *also different*, and the De Morgan property occurs in a trivial manner.

Regardless of this, we must make a distinction for the *common negations*, $N_k$, which establish the rotation of the atoms–once an order is chosen between them–displacing $k$ atoms in one direction or another.[97] All the other negations are called *exotic negations*. In the following sections, the reason for this distinction will be understood..

## Negations in **2Dn**

In what follows, two notations will be used for the lattice elements, as shown in Figure 13. We will refer to the *alphabetic notation*, as that which uses lower case letters for atoms and upper case for maximum elements. We will refer to the *mathematical notation*, as that which uses $d_i$ for atoms and y $D_i$ for maximum elements. The notation that uses letters in alphabetical order is simpler for truth tables, the notation that uses sub-indexes is useful in proving the properties of some functions.

The mathematical notation is used so that expressions in **2Dn** are easier to interpret. Therefore, rotations are expressed as $R_k\, d_i = d_{i+k}$ for *direct* rotations and $R_{-k}\, d_i = d_{i-k}$ for the *reverse* rotation.[98] Indexes are numbered $0, 1, \ldots, n-1$ y and all the operations are $n$-

[97] The existence of a rotation is suggested by the Greek elements and the double rotation by the Chinese elements.

[98] Naturally, the direction of rotation, either the reverse or direct, are conventional, this depends on the way how lattice elements are indexed and it is not absolute. For example, if the maximum element to the "right" of the atom of index 1 is 4, there would simply be a displacement constant in all the equations that follow, but the end result would not change.

module. Ultimately, it can be considered that $k$ has a sign. The rotation is similar for maximum elements. Unlike lattices of rank $r = 1$, these lattices have natural, well-defined rotations. This also occurs for ranks greater than 2.

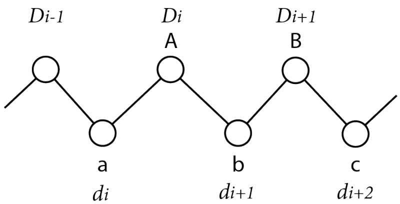


Figure 13: The two notations used in **2Dn**.

The systematic search for negation functions using a computer leads to $2n$ results which are expressed by two different lists, the list of *common* negations:[99]

- $N_0\, d_i = D_i \qquad N_0\, D_i = d_{i+1}$
- $N_1\, d_i = D_{i+1} \qquad N_1\, D_i = d_{i+2}$
- $\dots$
- $N_{n-1}\, d_i = D_{i+n-1} \qquad N_{n-1}\, D_i = d_i$

and the list of *exotic* negations, represented by $\tilde{N}$, is:

- $\tilde{N}_0\, d_i = D_{n-i} \qquad \tilde{N}_0\, D_i = d_{n-i}$
- $\tilde{N}_1\, d_i = D_{n-i+1} \qquad \tilde{N}_1\, D_i = d_{n-i+1}$
- $\dots$
- $\tilde{N}_{(n-1)}\, d_i = D_{n-i+n-1} = D_{-i-1} \qquad \tilde{N}_{(n-1)}\, D_i = d_{-i-1}.$

All the operations and numberings are $n$-module. These results are combined in the following theorems.

[99] When I had finished writing on this subject, Rafael Grompone suggested that I should add 1 to the indexes so that $N_{n-1}$ would go on to be $N_0$ and also to the lattice indexes. In this way, a more coherent and symmetrical nomenclature is obtained. Since this modification implies reviewing a great deal of material, this will be addressed in a later version.

**Theorem 34** *The* $n$ common *negations in* ***2Dn*** *correspond to the following equations (the operations are* $n$*-module):*
$N_k\, d_i = D_{i+k} \qquad N_k\, D_i = d_{i+k+1}.$

**Proof**. We intend to prove that these transformations comply with the De Morgan property. It is clear that the product of two different atoms is 0 and the sum of two maximum elements is 1. Then, $N_k(d_i \,.\, d_j) = N_k\, 0 = 1 = D_{i+k} + D_{j+k} = N_k\, d_i + N_k\, d_j$ and De Morgan is met. This is proven in a dual manner for the maximum elements. Given the sum of two non-contiguous atoms, it occurs that $N_k(d_i + d_j) = N_k\, 1 = 0 = D_{i+k} \,.\, D_{j+k} = N_k\, d_i \,.\, N_k\, d_j$ given that the maximum elements are also not contiguous. The same is valid in a dual manner for the maximum elements. If the atoms are contiguous, then $N_k(d_i + d_{i+1}) = N_k\, D_i = d_{i+k+1} = D_{i+k} \,.\, D_{i+k+1} = N_k\, d_i \,.\, N_k\, d_{i+1}$ and De Morgan is met. We still need to prove the case of a contiguous atom and maximum element. If $N_k(d_i \,.\, D_i) = N_k\, d_i = D_{i+k} = D_{i+k} + d_{i+k+1} = N_k\, d_i + N_k\, D_i$ and De Morgan is met. If considering the other case of contiguous elements $N_k(d_{i+1} \,.\, D_i) = N_k\, d_{i+1} = D_{i+k+1} = D_{i+k+1} + d_{i+k+1} = N_k\, d_{i+1} + N_k\, D_i$, it is also met. In the case of the sum, we have $N_k(d_i + D_i) = N_k\, D_i = d_{i+k+1} = D_{i+k} \,.\, d_{i+k+1} = N_k\, d_i + N_k\, D_i$, and it is met. Finally, for $N_k(d_{i+1} + D_i) = N_k\, D_i = d_{i+k+1} = D_{i+k+1} \,.\, d_{i+k+1} = N_k\, d_{i+1} + N_k\, D_i$, it is met. □

**Theorem 35** *The* exotic $n$ *negations in* ***2Dn*** *correspond to the following equations (the operations are* $n$*-module):*
$\tilde{N}_k\, d_i = D_{n-i+k} \qquad \tilde{N}_k\, D_i = d_{n-i+k}.$

**Proof**. The demonstration for exotic negations is done following along the lines of the previous theorem. In the case of the product of two different atoms, the sum of two different maximum elements or non-contiguous cases, De Morgan follows, just like before. For two contiguous atoms, the following occurs $\tilde{N}_k(d_i + d_{i+1}) = \tilde{N}_k D_i = d_{n-i+k} = D_{n-i+k-1} \,.\, D_{n-i+k} = \tilde{N}_k d_{i+1} \,.\, \tilde{N}_k d_i$ and De Morgan is met. For a contiguous atom and maximum element, if $\tilde{N}_k(d_i \,.\, D_i) = \tilde{N}_k\, d_i = D_{n-i+k} = D_{n-i+k} + d_{n-i+k} = \tilde{N}_k\, d_i + \tilde{N}_k\, D_i$, De Morgan is met. For the other contiguous case, the same thing occurs. In the case of the sum, we have $\tilde{N}_k(d_i + D_i) = \tilde{N}_k\, D_i = d_{n-i+k} = D_{n-i+k} \,.\, d_{n-i+k} = \tilde{N}_k\, d_i \,.\, \tilde{N}_k\, D_i$ and it is met. Similarly, for the other contiguous case it is also met. □

The application of a rotation to a common negation is an important case which is introduced in the following theorem.

**Theorem 36** *For common negations in **2Dn**, the following equations are valid: $R_k\, N_j = N_j\, R_k = N_{j+k}$. The operations are $n$-module and $k$ can be negative.*

**Proof**. Let us consider $R_k\, N_j\, d_i = R_k\, D_{i+j} = D_{i+j+k} = N_{j+k}\, d_i$. But $N_j\, R_k\, d_i = N_j\, d_{i+k} = D_{i+j+k}$, then this matches the previous result. For maximum elements something similar happens: $R_k\, N_j\, D_i = R_k\, d_{i+j+1} = d_{i+j+k+1} = N_{j+k}\, D_i$. But $N_j\, R_k\, D_i = N_j\, D_{i+k} = d_{i+j+k+1}$, ias in the previous case. The same property is valid in the case of a negative $k$ since the rotation equation is unique. □

As a result of this theorem, a common negation turns into another due to a rotation of the lattice elements, since the transformation equation $R_{-k}\, N_j\, R_k = N_j$ is valid. The product–successive application–of two common negations is a rotation.

**Theorem 37** *For exotic negations in **2Dn** the following equations are valid: $R_k\, \tilde{N}_j = \tilde{N}_j\, R_{-k} = \tilde{N}_{(j+k)}$. The operations are $n$-module and $k$ can be negative.*

**Proof**. The case of exotic negations is exactly the same. Let us consider $R_k\, \tilde{N}_j\, d_i = R_k\, D_{n-i+j} = D_{n-i+j+k} = \tilde{N}_{(k+j)}\, d_i$. But $\tilde{N}_j\, R_{-k}\, d_i = \tilde{N}_j\, d_{i-k} = D_{n-i+k+j} = \tilde{N}_{(j+k)}\, d_i$. The demonstration is the same for maximum elements. Then, it is proven. As in the previous case, $k$ can be negative. □

This result and the one obtained in **3Dn** do not contradict Theorem 19. There, it was established, for example, that $R_k\,\tilde{N}_j\,R_k^{-1}$ is a negation. In fact, $R_k\,\tilde{N}_j\,R_k^{-1} = \tilde{N}_j\,R_{-k}\,R_k^{-1} = \tilde{N}_j\,R_{-2k} = \tilde{N}_{(j+2k)}$.

Exotic negations are transformed in a special manner among themselves by the rotations. It is also interesting to consider the product of two negations.

**Theorem 38** *The product–successive application–of two negation in **2Dn** is, depending on the case:* $N_i\,N_j = R_{i+j+1}$, $\tilde{N}_i\,\tilde{N}_j = R_{i-j}$, $N_i\,\tilde{N}_j = S_{i+j+1}$ *and* $\tilde{N}_j\,N_i = S_{j-i}$. *The operations are* $n$*-module.*

**Proof.** Let us consider $N_i\,N_j\,d_k = N_i\,D_{j+k} = d_{i+j+k+1} = R_{i+j+1}\,d_k$. In the case $N_i\,N_j\,D_k = N_i\,d_{j+k+1} = D_{i+j+k+1} = R_{i+j+1}\,D_k$, then the first equation is met. In exotic negations it is: $\tilde{N}_i\,\tilde{N}_j\,d_k = \tilde{N}_i\,D_{n-k+j} = d_{n-(n-k+j)+i} = d_{k-j+i} = R_{i-j}\,d_k$ and $\tilde{N}_i\,\tilde{N}_j\,D_k = \tilde{N}_i\,d_{n-k+j} = D_{n-(n-k+j)+i} = D_{k-j+i} = R_{i-j}\,d_k$, then the second equation is met. Let us consider

$$N_i\,\tilde{N}_j\,d_k = N_i\,D_{n-k+j} = d_{n-k+j+i+1} = S_{i+j+1}\,d_k.$$

In the case of $N_i\,\tilde{N}_j\,D_k = N_i\,d_{n-k+j} = D_{n-k+j+i} = S_{i+j+1}\,D_k$, then the third equation is met. Let us now consider

$$\tilde{N}_j\,N_i\,d_k = \tilde{N}_j\,D_{k+i} = d_{n-(k+i)+j} = S_{j-i}\,d_k.$$

If we consider $\tilde{N}_j\,N_i\,D_k = \tilde{N}_j\,d_{k+i+1} = D_{n-(k+i+1)+j} = S_{j-i}\,D_k$, then the fourth equation is met. □

A consequence of this theorem is that $\tilde{N}_i\,\tilde{N}_i = R_0 = I$ where $I$ is the identity. Exotic negations are *involutory*. The structure of the group $\mathbf{G}_L$ of transformations of the lattice **2Dn** is made up of the $n$ rotations, the $n$ simmetries, the $n$ common negations and the $n$ exotic negations which are involutory.

**Theorem 39** *The product–successive application–of a negation and a simmetry in **2Dn** is, depending on the case:* $S_jN_k = \tilde{N}_{j-k-1}$, $N_kS_j = \tilde{N}_{j+k}$, $S_j\tilde{N}_k = N_{j-k-1}$ *and* $\tilde{N}_k\,S_j = N_{k-j}$. *All the operations are* $n$*-module.*

**Proof**. Let us consider

$$S_j N_k\, d_i = S_j D_{i+k} = D_{n-(i+k)+j-1} = \tilde{N}_{j-k-1} d_i.$$

Consideremos $S_j N_k D_i = S_j d_{i+k+1} = d_{n-(i+k+1)+j} = \tilde{N}_{j-k-1} D_i$, then the first equivalence is proven. If we consider $N_k S_j d_i = N_k\, d_{n-i+j} = D_{n-i+j+k} = \tilde{N}_{j+k}\, d_i$. If we consider

$$N_k S_j D_i = N_k\, D_{n-i+j-1} = d_{n-i+j-1+k+1} = \tilde{N}_{j+k}\, D_i,$$

then the second equivalence is proven. Let us consider

$$S_j \tilde{N}_k\, d_i = S_j D_{n-i+k} = D_{n-(n-i+k)+j-1} = D_{i-k+j-1} = N_{j-k-1}\, d_i.$$

If we consider

$$S_j \tilde{N}_k\, D_i = S_j\, d_{n-i+k} = d_{n-(n-i+k)+j} = d_{i-k+j} = N_{j-k-1}\, D_i,$$

then the third equivalence is proven. Let us consider

$$\tilde{N}_k\, S_j\, d_i = \tilde{N}_k\, d_{n-i+j} = D_{n-(n-i+j)+k} = D_{i-j+k} = N_{k-j}\, d_i.$$

If we now consider

$$\tilde{N}_k\, S_j\, D_i = \tilde{N}_k\, D_{n-i+j-1} = d_{n-(n-i+j-1)+k} = d_{i-j+k-1} = N_{k-j}\, D_i,$$

then the fourth equivalence is proven. □

This theorem completes the operations in the group $\mathbf{G}_L$ of automorphisms and negations of the lattice **L**.

An interesting observation to make is that there are common negations which are *involutory*. For this, it is enough to note that $N_i\, N_i = R_{2i+1}$. Then, the condition for $N_j$ to be involutory is that $2i + 1 = 0$ ($n$-module) and this has a solution for an uneven $n$. Therefore, for example, in **2D5** the negation $N_2 = (01)(aC)(bD)(cE)(dA)(eB)$ is involutory.

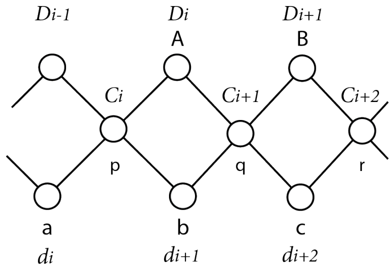


Figure 14: The two notations used in **3Dn**.

## Negations in **3Dn**

The Figure 14 shows the notation used in **3Dn** lattices as an extension of the **2Dn** case: the symbols $C_i$ are used for the central values–the *mathematical notation*–instead of the *alphabetic notation* $p, q, r, \ldots$ which is more compact for presenting the truth tables of the functions.

A software program performs a systematic search to find negations in **3Dn**. As in the previous case, these are divided in two groups of $n$ negations each: regular negations and exotic negations. Common negations in the **3Dn** case are:

- $N_0\, d_i = D_i \quad N_0\, C_i = C_{i+1} \quad N_0\, D_i = d_{i+2}$
- $N_1\, d_i = D_{i+1} \quad N_1\, C_i = C_{i+2} \quad N_1\, D_i = d_{i+3}$
- $\cdots$
- $N_{n-1}\, d_i = D_{i+n-1} \quad N_{n-1}\, C_i = C_i \quad N_{n-1}\, D_i = d_{i+n+1}$.

In summary, the general expression for common negations, in the general case, is:

$$N_j\, d_i = D_{i+j} \quad N_j\, C_i = C_{i+j+1} \quad N_j\, D_i = d_{i+j+2}$$

where all the operations are performed in $n$-module. The negation $N_{n-1}$ corresponds to a symmetry of the lattice around the central values and is involutory. Thus, for example, $N_{n-1}\, d_j = D_{n-1+j}$. If we apply once again $N_{n-1}\, D_{n-1+j} = d_{n-1+j+n-1+2} = d_{2n+j} = d_j$. The central values,$N_{n-1}\, C_i = C_{n-1+i+1} = C_i$, remain unaltered.

The exotic negations in the **3Dn** case are:

- $\tilde{N}_0\, d_i = D_{n-i} \quad \tilde{N}_0\, C_i = C_{n-i} \quad \tilde{N}_0\, D_i = d_{n-i}$
- $\tilde{N}_1\, d_i = D_{n-i+1} \quad \tilde{N}_1\, C_i = C_{n-i+1} \quad \tilde{N}_1\, D_i = d_{n-i+1}$
- $\cdots$
- $\tilde{N}_{(n-1)}\, d_i = D_{-i-1} \qquad \tilde{N}_{(n-1)}\, C_i = C_{-i-1}$
  $\tilde{N}_{(n-1)}\, D_i = d_{-i-1}.$

In summary, the general expression of the exotic negations is:

$$\tilde{N}_j\, d_i = D_{n-i+j} \quad \tilde{N}_j\, C_i = C_{n-i+j} \quad \tilde{N}_j\, D_i = d_{n-i+j}$$

where all the operations are performed as $n$-module. Of course, it is necessary to prove that these expressions meet De Morgan. This demonstration will be omitted since it does not contribute much to the one presented for **2Dn**.

**Theorem 40** *For common negations in **3Dn**, the following equations are valid: $R_k\, N_j = N_j\, R_k = N_{j+k}$. The operations are $n$-module and $k$ can be negative.*

**Proof**. Given $R_k\, N_j\, d_i = R_k\, D_{i+j} = D_{i+j+k} = N_{j+k}\, d_i$. Given $R_k\, N_j\, C_i = R_k\, C_{i+j+1} = C_{i+j+k+1} = N_{j+k}\, C_i$. Given $R_k\, N_j\, D_i = R_k\, d_{i+j+2} = d_{i+j+k+2} = N_{j+k}\, D_i$. If we consider the product backwards, we have that $N_j\, R_k\, d_i = N_j\, d_{i+k} = D_{i+j+k}$, as in the previous product. In the same way, it occurs that $N_j\, R_k\, C_i = N_j\, C_{i+k} = C_{i+j+k+1}$ and $N_j\, R_k\, D_i = N_j\, D_{i+k} = d_{i+j+k+2}$, as needed to be proven. □

This theorem shows that common negations are unchanged by the lattice rotations..

**Theorem 41** *The product–successive application–of two negations in **3Dn**, is, depending on the case: $N_i\, N_j = R_{i+j+2}$ and $\tilde{N}_i\, \tilde{N}_j = R_{i-j}$. The operations are $n$-module.*

**Proof**. Given $N_i\, N_j\, d_k = N_i\, D_{j+k} = d_{i+j+k+2} = R_{i+j+2}\, d_k$. Given $N_i\, N_j\, C_k = N_i\, C_{j+k+1} = C_{i+j+k+2} = R_{i+j+2}\, C_k$. Given

$N_i\, N_j\, D_k = N_i\, D_{j+k+2} = d_{i+j+k+2} = R_{i+j+2}\, d_k$, then it is proven for common negations. Given $\tilde{N}_i\, \tilde{N}_j\, d_k = \tilde{N}_i\, D_{n-k+j} = d_{n-(n-k+j)+i} = d_{k+i-j} = R_{i-j}\, d_k$. The same thing happens in the other cases since the transformations are equal, then, it is proven. □

Then, all exotic negations are invocatory–since $\tilde{N}_i\, \tilde{N}_i = R_0 = I$– as in **2Dn**. A common involutory negation also exists if $2i + 2 = 0$ ($n$-module).

As in the previous case, Theorem 39, is met, the demonstration is the same and is omitted.

## Overview of the group of automorphisms and negations

This section summarizes the different results obtained with regards to automorphisms and negations in **rDn** lattices. In Table 8 the results obtained in Theorems 2, 8, 9, 36, 37, 38 and 39, are repeated in an orderly fashion, as proven for **2Dn**. The result is understood to be the successive application of the row element and then of the column element.

Table 8: Successive application of automorphisms and negations.

| $\times$ | $R_k$ | $S_k$ | $N_k$ | $\tilde{N}_k$ |
|---|---|---|---|---|
| $R_j$ | $R_{j+k}$ | $S_{j+k}$ | $N_{j+k}$ | $\tilde{N}_{j+k}$ |
| $S_j$ | $S_{j-k}$ | $R_{j-k}$ | $\tilde{N}_{j-k-1}$ | $N_{j-k-1}$ |
| $N_j$ | $N_{j+k}$ | $\tilde{N}_{j+k}$ | $R_{j+k+1}$ | $S_{j+k+1}$ |
| $\tilde{N}_j$ | $\tilde{N}_{j+k}$ | $N_{k-j}$ | $S_{j-k}$ | $R_{j-k}$ |

The demonstrations extend naturally to **rDn** lattices by recurrence, as has been exemplified in some cases of **3Dn**.

# Penetration of opposites

## Introduction

Spontaneous dialectics and Hegel's attempt at formalization introduce two new notions which are foreign to binary logic: the *unity and struggle of opposites*–also called *penetration of opposites*–and the notion of *becoming*. These ideas have been intuitively introduced in the initial chapters. To binary logic, the existence of opposites evidences the falseness of a theory–they cannot coexist. The same happens with becoming: logical statements are eternal and unchangeable. Nothing can be true today and false tomorrow.

What are, then, the penetration of opposites and becoming? They are logical functions with two variables defined within dialectic lattices. These functions convey the unity and struggle of opposites and the negation of a negation, to use Hegelian language. They also describe, respectively, synchronic and diachronic opposites, while extending the meaning of binary logical functions. This means that, when 0 and 1 are applied to the values, they must only yield 0 or 1 as a result, that is to say, they must match known binary logical functions. This first condition can be referred to as the *principle of permanence of binary properties* or simply, the *principle of permanence*, abbreviated as PP.

A second consideration is valid. Given that dialectic lattices have rotations, as derived from the definition of dialectic lattices, it is necessary that the rotations in these dialectic functions be invariable. If this is not so, there would be privileged dialectic values, which would contradict the evidence of the symmetry generated by the rotations. This second condition is called *rotation invariance*, abbreviated as RI.

When searching for penetration functions, aside from the principles of permanence and rotation, we must consider the formal properties derived from the spontaneous application of these notions, as occurs in the different examples cited when we introduced natural dialectics.

## Overview on dialectic penetration

Let us now analyze some cases of penetration as used in the natural languages. This use, possible allows us to extract the formal properties they comply with from a dialectic standpoint. We will begin by adversative conjunctions.

Love sonnets show yet another aspect of dialectic penetration. We will consider Lope's statement, but introducing slight changes in presentation:

> to faint, to dare, to be enraged, coarse, tender, liberal, elusive, encouraged, mortal, dead, alive, loyal, traitor, coward, brave

Let us now formalize the penetrations by way of associations. It seems clear that the idea expressed is the following:

> (to faint, to dare, to be enraged), (coarse, tender), (liberal, elusive, encouraged), (mortal, dead, alive), (loyal, traitor), (coward, brave)

We can see that there are opposite *pairs*–such as (loyal, traitor) or (venturesome, repressed)–, but there are also *triads* of opposites, such as (to faint, to dare, to be enraged) or (mortal, dead, alive). In this example, it is undoubtable that logical penetration is *commutative*: the order of the terms does not matter in any of the cases. The analysis of the triple cases demands some additional considerations that are presented further ahead.

On the other hand, it also seems to be clear that in this text there might be two types of commas: some replace the dialectic penetration and the others–those that join pairs or triads of opposites–could either be an **AND** or an **OR** function, or even a commutative penetration. This matter calls for further analysis. If we believe they stand in place of **AND**, the phrase states that love contains *all* the contradictions, something which is truly exaggerated. If we believe them to stand for **OR**, the phrase states that it is enough to have *some* of the contradictions in the list in order to define love. However, it seems that neither **AND** nor **OR** are being conveyed, but a logical function that lies *somewhere* *in between the two*, one that is not as strong as **AND** or as loose as **OR**. This is why it also seems that the second type of commas replace a penetration, which is forcefully *commutative*, aside from *associative*.

Let us now examine the problem of contrary triads. For a triad to make sense, it is necessary that the penetration be–at least in some cases–*associative*; otherwise, it would not make sense.[100] Thus, for example, the order of the triad is not important and this calls for the two properties, commutative and associative, to be valid in the unity and struggle of opposites.

Intuitively, we can define the penetration as a function that meets the following properties:

- is associative (A) and commutative (C),
- meets the principle of permanence (PP) of binary properties,
- its elements are rotationally invariant (RI),
- is a dialectic function which is halfway between **OR** and **AND**,has the dialectic penetration property (DP).

This last property adds a new formal element. Given that $x\,.\,x = x + x = x$, we must add to this set of properties that the function is *idempotent* (I).

Table 9: General scheme of the penetration functions.

| | 0 | dialécticos | 1 |
|---|---|---|---|
| 0 | 0 | $f_1(y)$ | 0, 1 |
| dialécticos | $f_1(x)$ | $g(x,y)$ | $f_2(x)$ |
| 1 | 0, 1 | $f_2(y)$ | 1 |

[100] Lope's example also shows the *limits of the associative property*. It is difficult to accept this equality: (to faint, to dare, to be enraged), (coarse, tender), (liberal, elusive, encouraged), (mortal, dead, alive), (loyal, traitor), (coward, brave) = (to faint, to dare), (to be enraged, coarse), (tender, liberal, elusive), (encouraged, mortal), (dead, alive, loyal), (traitor, coward, brave). This suggests that the two types of commas are *different penetration functions*. This is clarified in what follows.

Table 9 presents the general scheme of a penetration function. The function $g(x, y)$ is the essential part of the function and has properties I, A, C and DP. The values corresponding to $0 * 0 = 0$ and $1 * 1 = 1$ are a consequence of idempotency. Functions $f_1, f_2$ also have properties I, A, C and DP, PP is evident.

The below shows that there are two types of penetration functions which we have referred to as *ample penetrations* and *strict penetrations*. Ample penetrations comply with the DP property for every pair of lattice elements and are important to the notion of *quantifiers*. Strict penetrations comply with the DP property only when the penetration of opposites is a thesis. This second type of penetrations is important, as are quantifiers, due to its connection to the notion of becoming of opposites.

## General property of dialectic penetrations

The generic property of the penetration function consists in having a logical value that is intermediate to the **AND** and **OR** functions. In order to define it, a set of formal and semantic properties must be met. The purpose of this section is to analyze these properties.

The starting point is a set of auxiliary theorems prior to the formal definition of the penetration function.[101] For this, it is necessary to introduce the notion of semi-lattice.[102]

---

[101] The theorems considered are known results, see Birkhoff [4, II, 3, Ex.1], but specialized on semi-lattices.

[102] In this case, a $\diamond$ operation is used, which has the formal properties of the sum within a lattice. A *dual* semi-lattice can also be defined by means of a $\odot$ operation that has the formal properties of a product. Performing a symmetry of the lattice's structure yields the same case.

**Definition 24** *A set of **S** elements is a* semi-lattice *if, for every pair $x, y$ of its elements, there is an operation $x \diamond y \in$ **S** with the properties:*

1. Idempotency *(I). It is met that $x \diamond x = x$.*
2. Associative *(A). It is met that $(x \diamond y) \diamond z = x \diamond (y \diamond z)$.*
3. Commutative *(C). $x \diamond y = y \diamond x$.*

In a semi-lattice with a *non-trivial* $\diamond$ operation[103] that has the properties I, A, C, a partial order can be defined (which justifies the name of semi-lattice).

**Theorem 42** *If $\diamond$ is a* non-trivial *operation, among elements of a set **S**, which has the properties I, A, C, then it is a partially ordered set which has the relation $x \leq y$ is defined as $x \diamond y = y$. The $+$ operation is defined by $x + y = x \diamond y$.*

**Proof**. The relation $x \leq y$ defined as $x \diamond y = y$ is an order relation because it meets: 1) idempotency due to property I; 2) if $x \leq y$ and $y \leq x$, then we have $x = x \diamond y = y$; 3) transitivity, because if $x \leq y$ and $y \leq z$ then $x \diamond y = y$, $y \diamond z = z$, but due to property A, then it occurs that $x \diamond z = x \diamond (y \diamond z) = (x \diamond y) \diamond z = y \diamond z = z$ then $x \leq z$. We still need to prove that if $z \geq x, z \geq y$ then $z \geq x + y = x \diamond y$ –that is, $z = z \diamond (x \diamond y)$–for $x + y$ to be the minimum upper limit. Due to the hypothesis, $z = z \diamond x, z = z \diamond y$ then $z = (z \diamond x) \diamond y = z \diamond (x \diamond y)$ due to A, as needed to be proven. □

The reciprocal theorem is also valid.[104]

[103] The operation is trivial if it meets that for every pair of elements $x \neq y$, $x \diamond y = a$ holds true, where $a$ is always the same element in the set.

[104] The dual theorems in which an operation $x \,.\, y = x \diamond y$ and the relation $x \leq y$ defined as $x \diamond y = x$. The demonstrations are done in a dual way. Regardless of this, in the applications *we will always consider the case of the sum.*

**Theorem 43** *If **S** is a semi-lattice for the* $+$, *operation, then the* $\diamond$ *operation defined* $x \diamond y = x + y$ *has the properties I, A, C.*

**Proof**. The demonstration follows immediately from this: 1) the operation $+$ is idempotent; 2) the operation $+$ is associative; 3) the operation $+$ is commutative. □

## Ample penetrations

Having established these results, it is now possible to define the penetration function in a general manner.

**Definition 25** *An* ample penetration function, *or simply* penetration, *refers to a binary operation in a lattice **L**, expressed as* $x * y$, *where* $x, y \in$ ***L**, complying with:*

1. *the* formal *properties I, A, C;*
2. *the* principle of permanence *of binary properties PP;*
3. rotational invariance *(RI): if* $x * y = z$ *then, if* $R$ *is a lattice rotation,* $R\,x * R\,y = R\,z$ *holds true.*
4. Dialectic penetration *(DP): two elements* $x, y$ *comply with* $x\,.\,y \leq x * y \leq x + y$.

The following theorem links the penetration function with the common negations.

**Theorem 44** *If* $*$ *is a penetration function in a dialectic lattice and* $N$ *is a common negation, then the function defined as* $*_n = N^{-1}(N\,x * N\,y)$ *also defines a penetration. This function is independent from the negation* $N$ *used.*

**Proof**. Let us consider the properties in order.

- I The function defined is idempotent given that $N^{-1}(N\,x{*}N\,x) = N^{-1}N\,x = x$.
- C It is commutative, given that $N^{-1}(N\,x * N\,y) = N^{-1}(N\,y * N\,x)$.
- A It is associative, given that $(N\,x{*}N\,y){*}N\,z = N\,x{*}(N\,y{*}N\,z)$ due to the A property of $*$. Introducing $N\,N^{-1}$ –the identity– we obtain $N\,N^{-1}(N\,x * N\,y) * N\,z = N\,x * N\,N^{-1}(N\,y * N\,z)$. By adding parentheses for purposes of clarity and applying $N^{-1}$ we have $N^{-1}(N\,(N^{-1}(N\,x * N\,y)) * N\,z) = N^{-1}(N\,x * N\,(N^{-1}(N\,y{*}N\,z)))$, which is the expression of the A property for the function defined.
- RI It is rotational invariant given that $N^{-1}(R\,N\,x{*}R\,N\,y) = N^{-1}R(N\,x{*} N\,y) = R\,N^{-1}(N\,x * N\,y)$ because common negations commute with $R$.
- DP $N\,x\,.\,N\,y \leq N\,x * N\,y \leq N\,x + N\,y$ holds true because of the DP property of $*$. By applying the negation $N^{-1}$ to these relations, we obtain $x + y \geq N^{-1}(N\,x * N\,y) \geq x\,.\,y$, which proves DP for the function defined.

Every common negation can be expressed as $N = N_0\,R^i$ and $N^{-1} = N_0\,R^{n-i}$. Then $N^{-1}(N\,x * N\,y) = R^{n-i}N_0^{-1}(R^i\,N_0\,x * R^i\,N_0\,y) = R^{n-i}N_0^{-1}\,R^i(N_0\,x * N_0\,y) = N_0^{-1}(N_0\,x * N_0\,y)$, given that the penetrations comply with RI, then the function defined is independent from the common negation used. This result depends on the commutative property of the common negations with the rotations. □

The ample penetration functions are also defined through the following definition.

**Definition 26** *The* ample penetration function $*$ *in the dialectic lattice **L** is defined as: for* $x, y < 1$ *then* $x * y = x\,.\,y$*, for every* $x$ *then* $x * 1 = 1 * x = 1$.
*The* ample penetration function $*_n$ *is defined as: for* $x, y > 0$ *then* $x *_n y = x + y$*, for every* $x$ *then* $x *_n 0 = 0 * n\,x = 0$.
*In an equal manner,* $*_n$ *can be defined by theorem 44.*

Definitions 25 and 26 match, as shown by the following thorem.

**Theorem 45** *The two definitions of ample penetrations, 25 and 26, are equivalent.*

**Proof**. Definition 26 complies with Definition 25. We can prove the properties in order:

- I holds true due to the idempotency of the sum and of the product.
- C holds true due to the commutative property of the sum and of the product and due to the explicit definitions in cases 0 and 1, respectively.
- A holds true due to the associative property of the sum and of the product. For case 1, if one or more of the intervening values is 1, the result is 1 no matter how they are associated; thus, for example, $x * (y * 1) = x * 1 = 1$ and $(x * y) * 1 = (x \,.\, y) * 1 = 1$. The other cases are similar and the same is valid for $*_n$ and 0.
- RI holds true because the sum and product are rotational invariant, so are 0 and 1.
- DP holds true by definition. Therefore, for example, $x \,.\, y \leq x * y = x + y \leq x + y$. For case 1, it occurs that $x = x \,.\, 1 \leq x * 1 = 1 \leq x + 1 = 1$.The same is valid for $*_n$ and 0.

Definition 25 complies with Definition 26. Let us now consider the semi-lattices in Figure 15 made up of elements 0 and 1 and the set **D** of their dialectic values or their negation ***N*D**. Because of its structure, it is a set that is partially ordered by an operation $\leq$ of the elements of the dialectic lattice.

According to Theorem 42 that refers to the properties of the functions with properties I, A, C, these functions are a sum in a semi-lattice defined by this penetration. Let us consider the case of each semi-lattice. We will analyze the properties in order:

- I: follows immediately, as do A and C, given that the sum has these properties.

RI: is valid because **D** is rotational invariant.

DP: holds true due to an adequate choice of the opposites, as can be seen in the examples that follow.

Then, the theorem is proven. □

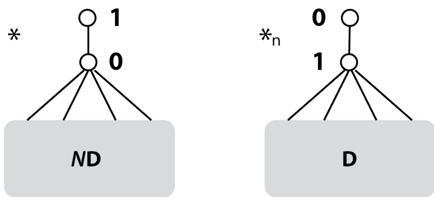


Figure 15: Semi-lattices for ample penetration functions.

Let us consider two penetration functions, $*'$, $*''$ in a lattice. With these, it is possible to build ample penetration functions–which do not meet the associative property–by means of the sum.

**Theorem 46** *The function defined as* $x * y = x *' y + x *'' y$ *–where* $*'$ *and* $*''$ *are penetrations that meet I, C, A, RI and DP–is an ample penetration complying with I, C, RI and DP but not A.*

**Proof**. Properties I, C follow immediately, given that both $*'$ and $*''$ comply. The same occurs with the RI property because each of them comply with it and the sum also meets RI. Given now $x, y$, the following inequalities are met:

$$x \,.\, y \leq x *' y \leq x + y \qquad x \,.\, y \leq x *'' y \leq x + y.$$

Given, for example,, $x \,.\, y \leq x *' y$ and $x \,.\, y \leq x *'' y$, from here $x \,.\, y \leq x *' y + x *'' y$ can be deduced due to the monotony of the sum. The other inequality is proven in a dual manner. The "sum" of penetrations does not meet A because in **3Dn** –see Table 12– $(a * b) * p = (0 + 0) * p = 0$ is met due to DP. Conversely, $a * (b * p) = a * (0 + p) = a * p = p + 0 = p$, with which it is proven by way of a counterexample. □

## Ample penetrations in **Dn**

Rank-1 penetration functions have very simple properties, something which does not occur in the higher ranks. A systematic search for the functions, aside from **AND**, **OR**, shows that only two functions are presented in Table 10.

As per Theorem 45, there are two semi-lattices in **Dn** whose sums generate these functions. Figure 16 illustrates them for case **D4**.

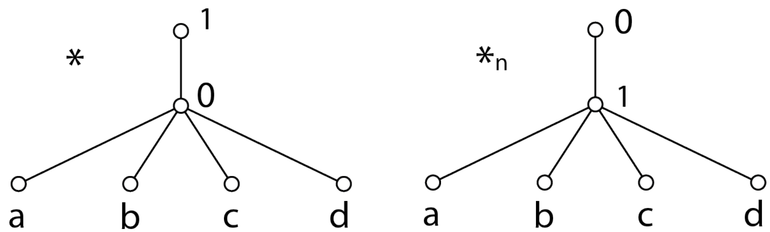


Figure 16: Semi-lattices for penetration functions in **D4**.

This diagram illustrates Theorem 44. If a negation $N$ is applied to one of the semi-lattices, the other is obtained. In fact, the four elements become themselves and values 0 and 1 are exchanged. These penetrations are symmetrical. The way in which the functions are generated allows us to make a generalization for lattices of a higher rank.

Table 10: Tables of dialectic penetrations in **D4**.

| $*$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| a | 0 | a | 0 | 0 | 0 | 1 |
| b | 0 | 0 | b | 0 | 0 | 1 |
| c | 0 | 0 | 0 | c | 0 | 1 |
| d | 0 | 0 | 0 | 0 | d | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| $*_n$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 1 | 1 | 1 | 1 |
| b | 0 | 1 | b | 1 | 1 | 1 |
| c | 0 | 1 | 1 | c | 1 | 1 |
| d | 0 | 1 | 1 | 1 | d | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |

## Ample penetrations in **2Dn**

Penetration functions in **2Dn** can be obtained from Theorem 45 and from the observations made for case **D4**. Figure 17 presents two semi-lattices which generate the penetration functions. These penetrations are symmetrical.

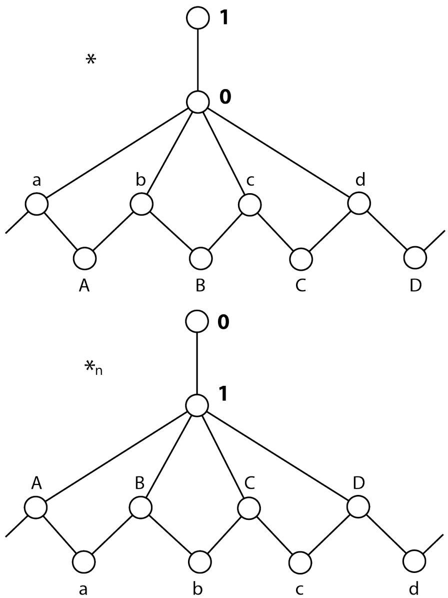


Figure 17: Semi-lattices for penetration functions in **2D4**.

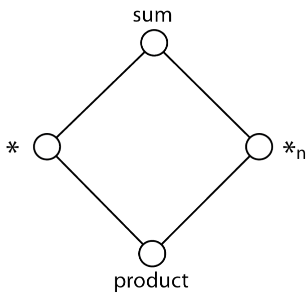


Figure 18: Diagram relating sum, product and penetrations.

A general observation must be made that applies to the penetrations analyzed. The relations established in the definition allow us to elaborate the diagram in Figure 18. The order relation lets us construct a lattice **D2** that establishes these relations. As immediately follows, this lattice—which builds a ying-yang dialectics—has a negation that transforms sum and product into themselves and also $*$ into $*_n$. It follows that Theorem 47 is met. These results are of interest in analyzing dialectic quantifiers. Please refer to the corresponding chapter.

Table 11: Tables of dialectic penetrations in **2D4**.

| $*$ | 0 | a | b | c | d | A | B | C | D | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| a | 0 | a | | | | a | | | a | 1 |
| b | 0 | | b | | | b | b | | | 1 |
| c | 0 | | | c | | | c | c | | 1 |
| d | 0 | | | | d | | | d | d | 1 |
| A | 0 | a | b | | | A | b | | a | 1 |
| B | 0 | | b | c | | b | B | c | | 1 |
| C | 0 | | | c | d | | c | C | d | 1 |
| D | 0 | a | | | d | a | | d | D | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| $*_n$ | 0 | a | b | c | d | A | B | C | D | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | A | 1 | D | A | 1 | 1 | D | 1 |
| b | 0 | A | b | B | 1 | A | B | 1 | 1 | 1 |
| c | 0 | 1 | B | c | C | 1 | B | C | 1 | 1 |
| d | 0 | 0 | 1 | C | d | 1 | 1 | C | D | 1 |
| A | 0 | A | A | 1 | 1 | A | 1 | 1 | 1 | 1 |
| B | 0 | 1 | B | B | 1 | 1 | B | 1 | 1 | 1 |
| C | 0 | 1 | 1 | C | C | 1 | 1 | C | 1 | 1 |
| D | 0 | D | 1 | 1 | D | 1 | 1 | 1 | D | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 11 presents truth tables for the corresponding penetration functions. This lattice-based generation is common to all **rDn** lattices. For the sake of clarity, zeros are omitted in dialectic values.

**Theorem 47** *Ample penetration functions meet the following:*

$$x * y = x \,.\, y + U(x,1) + U(y,1)$$

$$x *_n y = (x + y) \,.\, U(x,0) \,.\, U(y,0)$$

*where $U(x,a)$, and likewise for $y$, are the unit functions, Definition 21.*

**Proof**. Let us consider the case $*_n$. Due to the definition, it is clear that for 1 or for dialectic values, $x *_n y = x + y$ is valid. Conversely, for those same values, $x *_n 0 = 0$ is met, then it is clear that the second equality is met given that $N\,U(x, 0)$ is 1 for every $x \neq 0$ and 0 for $x = 0$. The same happens with $N\,U(y, 0)$. Then, the second equation is proven. The first is a consequence of the demonstrated, applied to $Nx$, $Ny$ and of Theorem 44 which leads to $x \,.\, y + U(Nx, 0) + U(Ny, 0)$ and is equivalent to the first equation. The first equation is proven. □

It follows that this theorem is valid in **Dn** and also in the general case, **rDn**, due to how penetration functions are constructed.

## Ample penetrations in **3Dn** and subsequents

Penetrations in **3Dn** and in more complex lattices follow the same scheme as the previous cases. Table 12 introduces penetration $*$. Penetration $*_n$ is obtained through transformation by means of any negation of the lattice, see Theorem 44. 0 values are omitted in the dialectic area.

Lattice **3D5** and subsequents also meet the condition of symmetrical penetrations. The diagram in Figure 18 and Theorem 47 are also met.

## Strict penetrations in **3Dn** and subsequents

Theorem 47 clearly shows that ample penetrations are only a slight modification of lattice operations. The intuitive notion of penetration seems to call for functions with more demanding properties. This occurs with strict penetrations, the topic of study of this section.

**3Dn** lattices are the simplest in which *strict* $\bar{*}$ penetration functions can be defined. This is a new penetration function which can be defined and its importance lies in its connection to the becoming function. These functions are generated via a new order in the dialectic elements.

We will analyze the **3D5** case as an example of the general function. Figure 29 presents the notation used–in this case, the alphabetical notation.

If we consider the values $a, p, A$ as the first case to be studied. We

Table 12: Truth table of penetration $*$ in **3D5**.

| $*$ | 0 | a | b | c | d | e | p | q | r | s | t | A | B | C | D | E | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| a | 0 | a | | | | | a | | | | a | a | | | a | a | 1 |
| b | 0 | | b | | | | b | b | | | | b | b | | | b | 1 |
| c | 0 | | | c | | | | c | c | | | c | c | c | | | 1 |
| d | 0 | | | | d | | | | d | d | | | d | d | d | | 1 |
| e | 0 | | | | | e | | | | e | e | | | e | e | e | 1 |
| p | 0 | a | b | | | | p | b | | | a | p | b | | a | p | 1 |
| q | 0 | | b | c | | | b | q | c | | | q | q | c | | b | 1 |
| r | 0 | | | c | d | | | c | r | d | | c | r | r | d | | 1 |
| s | 0 | | | | d | e | | | d | s | e | | d | s | s | e | 1 |
| t | 0 | a | | | | e | a | | | e | t | a | | e | t | t | 1 |
| A | 0 | a | b | c | | | p | q | c | | a | A | q | c | a | p | 1 |
| B | 0 | | b | c | d | | b | q | r | d | | q | B | r | d | b | 1 |
| C | 0 | | | c | d | e | | c | r | s | e | c | r | C | s | e | 1 |
| D | 0 | a | | | d | e | a | | d | s | t | a | d | s | D | t | 1 |
| E | 0 | a | b | | | e | p | b | | e | t | p | b | e | t | E | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

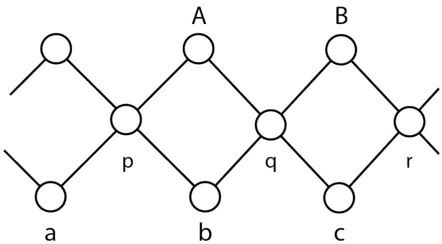


Figure 19: Simplified **3Dn** lattice diagram.

must try to make sense of the expression $x \,.\, y \leq x \mathbin{\bar{*}} y \leq x + y$, as long as $x \mathbin{\bar{*}} y$ is a thesis, and also that $x \mathbin{\bar{*}} y \neq 0$. Due to the semantics of penetration, it seems only natural to choose the definition $a \mathbin{\bar{*}} A = p$, given that $p$ is an *intermediate* value between $a$ and $A$.[105] After adopt-

[105] The Masonic triad–liberty, equality, fraternity–provides an interesting example of this notion. It is formed by two opposing elements, liberty and equality, and a synthesis, fraternity. It is clear that if human beings were truly fraternal, they would be free and equal.

ing this criterion, it naturally follows that $A \bar{*} p = p$, $a \bar{*} p = p$ and $p \bar{*} p = p$. On the other hand, $a \bar{*} a = a$ and $A \bar{*} A = A$ also occurs naturally. From these expressions it follows that for the triad $a, p, A$, $\bar{*}$ is commutative, associative and idempotent.

By the same token, by rotation we can obtain that the triads $b, q, B$ and those obtained by rotation are also commutative, associative and idempotent. Using identical reasoning, we can consider that $b, p, E$ or $c, q, A$ also generate a penetration function, albeit a different one. At this point we must also accept that the triads $b, p, A$ or $c, q, B$ present a different case, as well as $b, q, A$ or $c, q, B$. Ultimately, these considerations allow us to define new penetration functions different from common penetrations.

The previous considerations have left out the values 0 and 1. The idea behind these new penetration functions is to choose functions which only link theses formed by dialectic elements. In this manner, the following expressions $d \bar{*} 0 = 0 \bar{*} d = 0$, complemented each other, where $d$ is any given value. In actuality, the importance of strict penetrations lies mainly in the dialectic values, this is why the definition does not determine cases of $d \bar{*} 1 = 1 \bar{*} d$ which must be determined by additional semantic considerations. The new penetration functions operate strictly among dialectic theses, and for this reason, they can be referred to as *strict penetrations*–they do not involve non-dialectic values. These considerations lead to the following definition.

**Definition 27** *The* strict penetration $\bar{*}$ *will be referred to as a two-variable function in a dialectic lattice that has the following properties:*

1. $\bar{*}$ *has the properties I, A, C, RI;*
2. *binary penetration (PB) property: if $x$ is a lattice element $x \bar{*} 0 = 0 \bar{*} x = 0$ is met but no semantic conditions are defined for $x \bar{*} 1 = 1 \bar{*} x$;*
3. *dialectic penetration (PD) property: if $x, y$ meet that $x \bar{*} y$* is a thesis, *then the following is met: $x \,.\, y \leq x \bar{*} y \leq x + y$.*

It is worth noting that condition 2 is compatible with the properties I, C, RI. By definition, I and C follow. The associative property has several cases which are all proven in a similar manner. Let us consider a couple of representative cases: $(0 \bar{*} 1) \bar{*} x = 0 \bar{*} (1 \bar{*} x) = 0$, $(x \bar{*} y) \bar{*} 0$ is separated into two distinct cases 1) if $z = x \bar{*} y$ is a thesis $z \neq 0$ then it is worth $z \bar{*} 0 = 0$, 2) if $x \bar{*} y = 0$, then it is also worth 0.

If $x \bar{*}^0 1 = 0$, $x \bar{*}^d 1 = x$ or $x \bar{*}^1 1 = 1$, are used as additional semantic conditions in Definition 27, where $x$ is any given lattice value, the properties of Definition 27 are verified.[106]

Strict penetrations can also be obtained as sums in a semi-lattice, as shown in Figure 20, where **cD** is an appropriate arrangement of the lattice's dialectic elements, as shown in Figures 22 and 23.

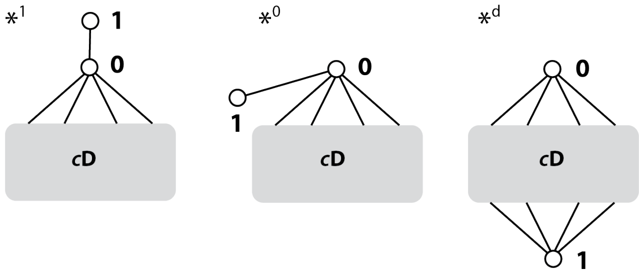


Figure 20: Semi-lattices for strict penetration functions.

> **Theorem 48** Strict penetrations *in **3Dn** lattices comply with the expressions in Table 13 except for the value 1.*

**Proof**. As per Theorem 42, these functions–which meet I, A, C–are sum operations in semi-lattices which must meet the DP condition. Figure 21 presents the schemes that allow us to write these functions. For purposes of meeting the DP condition, opposites–atoms and maximum elements–that keep the function different from 0 must meet

[106] It is worth noting that a result of the type $x * 1 = Rx$ does not meet the DP property. Thus, for example, $a = a \,.\, 1 \nleq a \bar{*} 1 = b = Ra$ and similarly for $R^i$. The case $x \bar{*} 1 = 1$ does not appear to have any applications of interest.

some of the conditions of the figure, they need to be adjacent. In this case, the conditions of opposites are met:

$$N_0\, d_i = D_i \quad N_{n-1} d_i = D_{i-1} \quad N_{n-1} D_i = d_{i+1} \quad N_{n-2}\, d_{i+1} = D_{i-1}$$

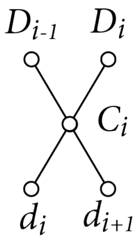


Figure 21: Diagram for strict penetration functions in **3Dn**.

Table 13 contains the generic expressions for the four strict penetrations. The idempotency equations–of the type $D_i \,\bar{*}_j D_i = D_i$ and others for atoms and central elements–have been omitted, as well as 0 values, since they are common to all functions. Functions involving elements 0 and 1 clearly are missing. There are four strict penetration functions. □

Table 13: Expressions of penetration functions in **3Dn**

| | | |
|---|---|---|
| $d_i \,\bar{*}_1 D_i = C_i$ | $d_i \,\bar{*}_1 C_i = C_i$ | $D_i \,\bar{*}_1 C_i = C_i$ |
| $d_i \,\bar{*}_2 D_{i-1} = C_i$ | $d_i \,\bar{*}_2 C_i = C_i$ | $D_{i-1} \,\bar{*}_2 C_i = C_i$ |
| $d_{i+1} \,\bar{*}_3 D_i = C_i$ | $d_{i+1} \,\bar{*}_3 C_i = C_i$ | $D_i \,\bar{*}_3 C_i = C_i$ |
| $d_{i+1} \,\bar{*}_4 D_{i-1} = C_i$ | $d_{i+1} \,\bar{*}_4 C_i = C_i$ | $D_{i-1} \,\bar{*}_4 C_i = C_i$ |

Table 14 presents the truth table for the dialectic penetration functiona $\bar{*}_1$ for this lattice. In our presentation 0 values have been omitted in the dialectic area for the sake of clarity. The function thus defined meets the properties I, A, C, RI, BP and DP.

Figure 22 shows the four schemes of semi-lattices generated by the functions. The value 1 has been omitted since it has a separate treatment in the semi-lattice, as shown in Figure 20.

Strict penetrations *are not ample penetrations*. Let us consider, for instance, $\bar{*}_1$. Es claro que $b \,\bar{*}_1 A = 0$ and that $b \,.\, A = b$, then, the PD property is met.

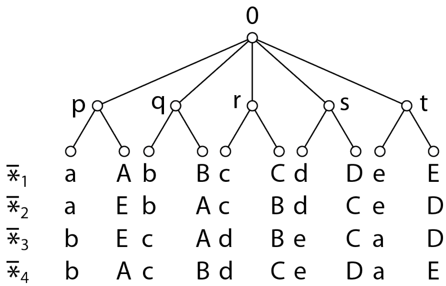


Figure 22: Semi-lattices for penetration functions in **3D5**.

Table 14: Truth table for strict penetration 1 in **3D5**.

| $\bar{*}_1^0$ | 0 | a | b | c | d | e | p | q | r | s | t | A | B | C | D | E | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | | | | | p | | | | | p | | | | | 0 |
| b | 0 | | b | | | | | q | | | | | q | | | | 0 |
| c | 0 | | | c | | | | | r | | | | | r | | | 0 |
| d | 0 | | | | d | | | | | s | | | | | s | | 0 |
| e | 0 | | | | | e | | | | | t | | | | | t | 0 |
| p | 0 | p | | | | | p | | | | | p | | | | | 0 |
| q | 0 | | q | | | | | q | | | | | q | | | | 0 |
| r | 0 | | | r | | | | | r | | | | | r | | | 0 |
| s | 0 | | | | s | | | | | s | | | | | s | | 0 |
| t | 0 | | | | | t | | | | | t | | | | | t | t0 |
| A | 0 | p | | | | | p | | | | | A | | | | | 0 |
| B | 0 | | q | | | | | q | | | | | B | | | | 0 |
| C | 0 | | | r | | | | | r | | | | | C | | | 0 |
| D | 0 | | | | s | | | | | s | | | | | D | | 0 |
| E | 0 | | | | | t | | | | | t | | | | | E | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 15: Truth table of strict penetration 2 in **3D5**.

| $\bar{*}_2^d$ | 0 | a | b | c | d | e | p | q | r | s | t | A | B | C | D | E | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | | | | | p | | | | | | | | | p | a |
| b | 0 | | b | | | | | q | | | | q | | | | | b |
| c | 0 | | | c | | | | | r | | | | r | | | | c |
| d | 0 | | | | d | | | | | s | | | | s | | | d |
| e | 0 | | | | | e | | | | | t | | | | t | | e |
| p | 0 | p | | | | | p | | | | | | | | | p | p |
| q | 0 | | q | | | | | q | | | | q | | | | | q |
| r | 0 | | | r | | | | | r | | | | r | | | | r |
| s | 0 | | | | s | | | | | s | | | | s | | | s |
| t | 0 | | | | | t | | | | | t | | | | t | | t |
| A | 0 | | q | | | | | q | | | | A | | | | | A |
| B | 0 | | | r | | | | | r | | | | B | | | | B |
| C | 0 | | | | s | | | | | s | | | | C | | | C |
| D | 0 | | | | | t | | | | | t | | | | D | | D |
| E | 0 | p | | | | | p | | | | | | | | | E | E |
| 1 | 0 | a | b | c | d | e | p | q | r | s | t | A | B | C | D | E | 1 |

Clearly, the value of the penetration function of two opposites has to be a central element for properties C and DP to be met. According to this condition, the expressions of the four penetration functions can be constructed.

Table 15 presents the truth table for penetration $\bar{*}_2$ where the option $x \,\bar{*}_2\, 1 = x$ has been chosen. This function has interesting properties related to functions of becoming and quantifiers. This is of interest in applications of the experimental and social sciences, as we will see in upcoming chapters.

Strict penetrations fail to meet Theorem 44 given that in the case of $\bar{*}_i^d$ we have that $a \,\bar{*}_i^d\, b = 0$ but $N_0\, a \,\bar{*}_i^d\, N_0\, b = A \,\bar{*}_i^d\, B = 0 \neq N_0\, 0$. The same thing occurs in the case of $\bar{*}_i^0$ or $\bar{*}_i^1$.

The study of strict penetration functions in higher-rank lattices does not appear to be of much interest, at least in the current state of this inquiry. In spite of this, it is possible to indicate how we can extend the penetration functions by means of a construction similar to the one shown in Figure 22.

The construction of strict penetration functions can continue in *odd-rank* lattices.As an example, we can consider the construction in **5Dn**, see Figure 23. In the figure, the value 1 has been omitted, which may occupy any of the relations in Figure 20.

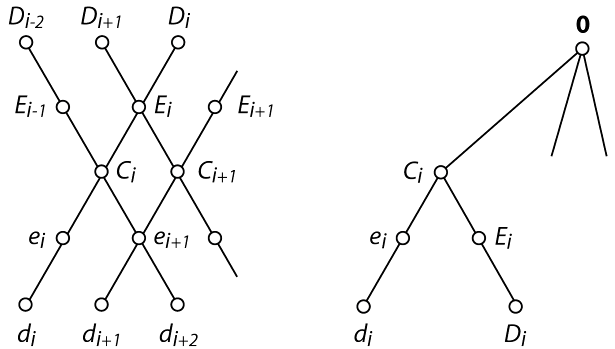

Figure 23: Lattice and semi-lattice for penetrations in **5Dn**.

The diagram on the left contains the mathematical notation for the lattice. The diagram on the right contains the semi-lattice that constructs one of the penetration functions. The other functions are constructed in the same manner, but with the elements $d_{i+2}$, $e_{i+1}$, $E_{i-1}$, $D_{i-2}$, etc.

Rank-5 strict penetrations allow us to make an in-depth study of some issues of class in the social sciences. For the time being, there do not seem to be any examples calling for a rank-7 lattice.

# Becoming

## Introduction

The second dialectic function that we must introduce describes the property of *becoming*, of the transformation. It is comprised by a two-variable function $f(x, y)$ that states that the dialectic value $x$ becomes–is transformed–into the dialectic value $y$. Therefore, we note it as $x \to y$.

The composition of the becoming function is based on identifying formal properties extracted from examples of becoming as performed by spontaneous human thought. The examples previously studied allow us to formalize it. They originate from human experience rather than from *a priori* speculation.

The first of the becoming properties states that immobility is impossible, as Heraclitus argued to Zeno. This means that the expression $a \to a$ is absolutely false.

The different examples introduced, which have originated from spontaneous thought throughout the centuries, show that there is a close relationship between the becoming function and negation. In most of the cases analyzed it appears in the following manner:

$$\cdots a \to N\,a \to N\,N\,a \to N\,N\,N\,a \to \cdots$$

where $a$ is a dialectic value and $N$ is a specific negation. This is almost always a closed process in a cycle which returns to the point of departure.

Naturally, a semantic property runs through the entire structure of the dialectics, the rotation invariance, RI, of the lattice. It is standard to think that the logical function of becoming fulfills this property, as a consequence of the commutation between the rotation and the negations.

A specific case of becoming involves the binary values 0 and 1. The usual interpretation of traditional logic tells us that $0 \to 0$ and $1 \to 1$

are *absolutely true*. Statements in logic or mathematics are considered *immutable*. As a result, $0 \to 1$ and $1 \to 0$ are *absolutely false*. By extension, a dialectic value cannot become 0 or 1 and reciprocally. These properties can be determined by a formal definition.

**Definition 28** *The becoming function, $x \to y$, is a two-variable function in a dialectic lattice which meets the following properties:*

1. Becoming of movement *(DM): If $d$ is a dialectic value, then $d \to d = 0$.*
2. Becoming of negation *(DN). If $d$ is a dialectic value and $N$ is any given negation, then $d \to N\,d$ is a thesis.*
3. Becoming of rotation *(DR). If $a \to b$ is a thesis, then $R\,a \to R\,b$, where $R$ is a lattice rotation, is also a thesis.*
4. Permanence of the formal values *(PP): True and false values are immutable, that is $0 \to 0 = 1$ and $1 \to 1 = 1$. As a complement, if $d$ is a dialectic value, then $d \to 0 = 0$, $d \to 1 = 0$, $0 \to d = 0$ and $1 \to d = 0$.*

Figure 16 presents the general structure of a becoming function, where $f(x, y)$ meets the properties BM, BN and BR. This dialectic function is the natural extension of the *equivalence* function in binary logic; it is enough to take a look at the relations between 0 and 1.

Table 16: Structure of the generic dialectic becoming.

| $\to$ | 0 | dialectic | 1 |
|---|---|---|---|
| 0 | 1 | 0 $\cdots$ 0 | 0 |
| dialectic | 0 | $f(x, y)$ | 0 |
| 1 | 0 | 0 $\cdots$ 0 | 1 |

Unlike other dialectic functions, where the result of the operation is what really matters, in the becoming function the result *usually bears* *little importance*–the chains formed, that have thesis value and describe the process of becoming, are the truly important elements. In order to continue with the analysis of becoming it is necessary to define what is understood by a chain of relations in becoming.

> **Definition 29** *The chain $a \to b \to c \to \cdots \to g \to h$ refers to the set of expressions in becoming, chained as a sequence between each other, which are* theses*: $a \to b$, $b \to c$, $\ldots$, $g \to h$.*

A general property of the becoming function is that every chain in becoming returns to the starting value.

> **Theorem 49** *If we consider the chain $a \to N\,a \to N\,N\,a \cdots$ where $a$ is a dialectic value and $N$ is* any *given negation, the chain goes back in itself: it is a* closed cycle.

**Proof**. There is a value of $p$ such that $N^p = I$, the identity, then, it returns to $a$. □

Naturally, involutory negations exist, and there are also chains comprised of only two elements, as several examples show. We can also observe that the becoming function *cannot be transitive*, because if from $a \to b \to c$ we could obtain that $a \to c$ is a thesis, then, each closed chain would prove that $a \to a$ is a thesis, contradicting the BM property of the becoming functions.

There is a general way of constructing a becoming function, as established by the following theorem.

> **Theorem 50** *The function defined in a dialectic lattice of rank $r > 1$ and an* ordinary *negation $N$, in which there is no $x$ such that $Nx = x$ (non-idempotent negation), is a becoming function if it meets the following conditions: 1) for each dialectic value $d$ and negation $N$, $d \to N\,d$ is a thesis, 2) for the remaining pairs of dialectic values it is worth 0 and 3) it meets PP.*

**Proof**. In $r > 1$ and $N$, due to 1), there is no $a$ such that $N\,a = a$, then, BM is met. The BN property is met due to 2). If $d \to N\,d = e$ is a thesis, then $R\,d \to R\,N\,d = R\,d \to N\,R\,d$ is a thesis due to the commutative property of ordinary negations and 1), then, BR is met. PP is met due to 3), then, it is proven. In the demonstration, the dialectic value $e$ does not participate.[107] □

This result shows that there are several becoming functions if we only take into account the chains, and not just the values that the becoming function acquires. It is worth noting that if $a \to b$ because $b = N\,a$, then, it also occurs that $b \to a$ because $a = N^{-1}b$ but these are two *different becoming functions*, unless $N = N^{-1}$, an involutory negation, takes place.

It is important to note that for exotic negations, the equality $R\,\tilde{N}_j = \tilde{N}_j\,R_{-1}$ is met, and the BR property is not.

## Becoming in **Dn**

According to the theorem of negations in **Dn**, several becoming functions can be constructed by means of negations $N_1, N_2, \ldots$ as well as by the choice of values that the function acquires. Table 17 presents the truth tables of two becoming functions, the first constructed from $N_1$ and the second, from $N_2$.[108] Zeros are omitted in the dialectic area.

Table 17: Some truth tables of becoming in **D4**.

| $\to$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | | b | | | 0 |
| b | 0 | | | c | | 0 |
| c | 0 | | | | d | 0 |
| d | 0 | a | | | | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |

| $\to$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | | | a | | 0 |
| b | 0 | | | | b | 0 |
| c | 0 | c | | | | 0 |
| d | 0 | | d | | | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |

[107] A generic equation for defining a becoming function is, for example, the expression $d \to N_i\,d = R_j\,d$ for the appropriate values of $i, j$.

[108] The equations used in the functions in Table 17 are, respectively: $d \to N_1\,d = R\,d$ and $d \to N_2\,d = d$. There are other possible examples–where the values are changed in the dialectic area–which meet the general properties of becoming.

In the first function, the closed cycle $a \to b \to c \to d \to a$ occurs, and in the second function, the cycles $a \to c \to a$ and $b \to d \to b$. This depends on the negations used in the lattice. In general, if $n$ is not a prime number, there are cycles which have a number of elements that are divisors of n, as 2 is a divisor of 4, in this case. In **D6**, for instance, there are cycles of 2, 3 and 6 elements.

The elements in the sequence of becoming are diachronic opposites and, to a large extent, they are equivalent among themselves. This is discussed below in more depth, through examples.

## Becoming in **2Dn**

The general theorem allows us to construct becoming functions in **2Dn**. With the aim of illustrating the method, we will choose case **2D4** using the notation from Figure 13. Table 18 presents the truth table constructed with the equation $d \to D = Rd$. Zeros are omitted in the dialectic area.

Table 18: Truth table of a becoming function in **2D4**.

| $\to$ | 0 | a | b | c | d | A | B | C | D | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | | b | | | b | | | | 0 |
| b | 0 | | | c | | | c | | | 0 |
| c | 0 | | | | d | | | d | | 0 |
| d | 0 | a | | | | | | | a | 0 |
| A | 0 | | B | | | | B | | | 0 |
| B | 0 | | | C | | | | C | | 0 |
| C | 0 | | | | D | | | | D | 0 |
| D | 0 | A | | | | A | | | | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

As can be constructed, the cycle $a \to A \to b \to B \to c \to C \to d \to D \to a$, is generated, but also cycles $a \to b \to c \to d \to a$ and $A \to B \to C \to D \to A$.

## Becoming in **3Dn**

As in all odd ranks, rank-3 lattices have *central elements*. This fact endows the becoming function with special properties. In addition, these lattices are very important with regards their application to the dialectics of history.

The general theorem allows us to construct becoming functions in **3Dn**. In order to illustrate the method, we will choose case **3D5** with the notation from Figure 14. $x \rightarrow N_0\, x = x$ is taken as the defining function. There is no point in writing down the truth table which is almost exclusively populated with zeros. For this reason, we prefer to only write down the significant dialectic values.

Table 19: A becoming function in **3D5**.

| | | |
|---|---|---|
| $a \rightarrow A = a$ | $p \rightarrow q = p$ | $A \rightarrow c = A$ |
| $b \rightarrow B = b$ | $q \rightarrow r = q$ | $B \rightarrow d = B$ |
| $c \rightarrow C = c$ | $r \rightarrow s = r$ | $C \rightarrow e = C$ |
| $d \rightarrow D = d$ | $s \rightarrow t = s$ | $D \rightarrow a = D$ |
| $e \rightarrow E = c$ | $t \rightarrow p = t$ | $E \rightarrow b = E$ |

This becoming function generates two causal cycles, the first between atoms and maximum elements

$$a \rightarrow A \rightarrow c \rightarrow C \rightarrow e \rightarrow E \rightarrow b \rightarrow B \rightarrow d \rightarrow D \rightarrow a$$

the second, among central elements

$$p \rightarrow q \rightarrow r \rightarrow s \rightarrow t \rightarrow p.$$

Despite having the same elements, the causal cycle of maximum elements and atoms is different from those found in **2D5**.

## Heraclitus' river

In **rD**$\infty$ lattices, the becoming function does not configure a closed causal cycle. Quite to the contrary, the successive negations always lead to new dialectic values. This is the logical model of Heraclitus' river: see page 58.

This model also applies to natural selection. The process of becoming of the species never returns to a previous stage, at least in the current state of our knowledge of nature. By extension, this also appears to be the logical model of the progress of human history, at least according to the materialistic interpretation, and not to Vico.

## Synchronic and diachronic opposites

The penetration and becoming functions are closely related to the synchronic and diachronic opposites described in the initial chapters. These functions allow us to formalize their definition and properties.[109]

**Definition 30** *Two elements, $x$ and $y$, different between themselves, in a dialectic lattice* **L** *are referred to as* synchronic opposites *if $x \bar{*} y$ is a thesis for a strict penetration function of the lattice. Two elements are referred to as* diachronic opposites *if $x \to y$ is a thesis for a becoming function of the lattice.*

This situation can be further exemplified in **3Dn** where the two types of opposites can be found and related within the same lattice. Figure 24 presents the lattice with the mathematical notation.

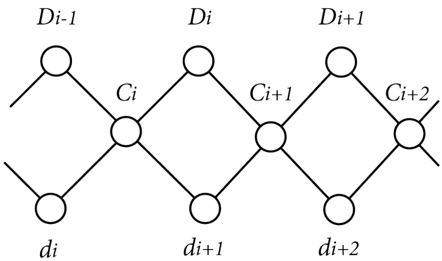


Figure 24: The **3Dn** lattice in mathematical notation.

These definitions allow us to prove a result which is of relevance in applying dialectics to various cases of interest. Let us consider the

[109] Note that if $x \bar{*} y$ is a thesis, the elements $x$ and $y$, different between themselves, are opposites since there is a negation that links them, see truth tables. In the case $x \to y$ they are opposites due to the definition of $\to$.

penetration $\bar{*}_1$ which is commutative.

> **Theorem 51** *In every **3Dn** lattice, the following equations are met:* $d_i \bar{*}_1 D_i \to C_{i+1}$, $C_i \to d_{i+1} \bar{*}_1 D_{i+1}$ *and* $d_i \bar{*}_1 D_i \to d_{i+1} \bar{*}_1 D_{i+1}$ *for all the values of* $i$.

**Proof**. It immediately holds true, see Table 13.

$$d_i \bar{*}_1 N_0\, d_i = C_i \qquad d_{i+1} \bar{*}_1 N_0\, d_{i+1} = C_{i+1} \qquad \cdots$$

Then, the following pairs of elements

$$d_i, N_0\, d_i = D_i \qquad d_{i+1}, N_0\, d_{i+1} = D_{i+1} \qquad \cdots$$

are synchronic opposites. On the other hand, we have that

$$C_{i+1} = N_0\, C_i \quad C_{i+2} = N_0\, q \quad \cdots$$

which indicates that we can write

$$C_i \to C_{i+1} \qquad C_{i+1} \to C_{i+2} \qquad \cdots$$

and therefore, these are diachronic opposites. As a consequence, holds true; by replacing elements for the results obtained, we obtain the equations that needed to be proven. □

This theorem is proven similarly for penetration 2.

> **Theorem 52** *In every **3Dn** lattice, the following equations are met* $d_i \bar{*}_2 D_{i-1} \to C_i$, $C_{i+1} \to d_{i+1} \bar{*}_2 D_i = C_i$, $d_i \bar{*}_2 D_{i-1} \to d_{i+1} \bar{*}_2 D_i$ *for all the values of* $i$.

**Proof**. The demonstration is similar to the previous case, see Table 13.

$$d_i \bar{*}_1 N_{n-1}\, d_i = C_i \qquad d_{i+1} \bar{*}_1 N_{n-1}\, d_{i+1} = C_{i+1} \qquad \cdots$$

Then, the following pairs of elements

$$d_i, N_{n-1}\, d_i = D_{i-1} \quad d_{i+1}, N_{n-1}\, d_{i+1} = D_i \quad \cdots$$

are synchronic opposites. On the other hand, we have that

$$C_{i+1} = N_0\, C_i \quad C_{i+2} = N_0\, q \quad \cdots$$

which indicates that we can write down

$$C_i \to C_{i+1} \qquad C_{i+1} \to C_{i+2} \quad \cdots$$

and therefore, these are diachronic opposites. As a consequence, holds true; by replacing elements of the results obtained, we obtain the equations that needed to be proven. □

Although this theorem appears to be similar to the previous, it has one substantial difference. In the first case, synchronic and diachronic opposites occur through negation $N_0$. In the second case, synchronic opposites occur through negation $N_{n-1}$ and diachronic opposites through negation $N_0$. The fact that only one negation takes part in the first theorem, while two are involved in the second, is essential to its application in the social sciences.

It is easy to extend these results to more complex lattices with an odd $r$. The application of these equations is important in the social sciences, as analyzed in the final chapter of this book.

# Argumentation

## Introduction

The word dialectic comes from the Greek διαλεκτικος (dialektikos), which means "pertaining to dialogue". The term gave way to the Latin *dialectice*, where it acquired the sense of telling true from false. The term subsequently made its way to the modern European languages. In the last centuries, it incorporated the meaning of analyzing opposites. The present chapter examines this idea of dialogue and resolution of contradictions. It is expected that a formalization of dialectics will allow us, at the very least, to "reason", regardless of the meaning endowed to the term.

Without question, the deductive model introduced by Euclid and formalized by George Boole, Gottlob Frege (1848, 1925) and Bertrand Russell has proven to be very successful. However, it fails to cover all aspects of the "natural reasoning" that we, as humans, perform.

For example, the British empiricists introduced *induction* as an additional mechanism for creating knowledge. The formal aspects of this methodology, which is associated with the natural sciences, aroused suspicion from early on. What was the formal structure of this manner of generating knowledge? For a couple of centuries, the topic remained in a haze. In the $20^{th}$, Karl Popper (1902, 1994) finally made a major breakthrough.

Popper's proposal was based on mathematics: statements–especially those derived from experience–are not proven, *but refuted*. By reversing the terms, the problem seemed to have been solved. The model followed by Popper has long been used in mathematics. The existence of a *counterexample* is sufficient proof of the falseness of a statement. Finding a case that cannot be proven is enough to refute the statement, dismiss it and ascertain its falseness.

For Popper–and all scientists–a statement has to be able to be refuted in one way or another. Maybe not right now, for technological or economic reasons, but it still bears the potential. If it cannot be refuted, it is either a philosophical or a theological statement, or it is a belief.

This works very well in mathematics and binary logic, where a statement can only be true or false. But it is not the case in dialectics, where there are many intermediate values of truth. We will need to take over this issue exactly where Popper left off.

## Argumentation

The theory of refutation has played a key role in illustrating the foundations of science. However, it proves to be quite partial in developing scientific statements.

Science is not cut and dried. It is not a set of precise statements awaiting potential refutation. Most outcomes in science are works in progress and only a few statements meet Popper's ideal conditions. How do scientists, then, go about their work using imprecise statements? When a new thesis is established, a quest for refutable statements begins–those that cannot be refuted by all that is known up to that moment. This Popperian property is not enough. While it is true that new theories must be useful, old, refuted theories can be useful as well. Table 20 presents some examples that we will analyze in this section.

Table 20: Examples of theories.

| | **useful** | **useless** |
|---|---|---|
| **refuted** | Newton's Mechanics | Aristotle's Mechanics |
| **not refuted** | Quantum Mechanics<br>Relativistic Mechanics | non-existing dinosaurs<br>non-expanding universe |

It seems clear that Newton's mechanics have refuted Aristotle's mechanics and rendered them useless, see [8]. While relativistic and quantum mechanics have refuted Newton's mechanics, their coexistence in different fields of application means the latter was not invalidated by the former. This table considers possible statements about the existence of dinosaurs and the expanding universe.

- Dinosaurs have never existed as living beings. What exist are fossilized bones formed as a geological phenomenon of the Earth's crust.[110]

- The universe is not expanding. The red shift observed by astronomers is due to *a property of telescopes*. When the light of a galaxy that lies far away travels through the telescope, it suffers a shift. This does not occur with lights originated at distances closer than one light-year.

While these statements are difficult to refute, it is not impossible to do so.[111] However, no scientist would even bother.

Why, then, do we reject these statements? We do, because, aside from the possibility of being refuted, scientific statements must have *a justifying argument*. A true scientist will demand *a reason* for which the light coming from a distant origin to behave differently as it travels through a telescope; he will demand to know the exact make and model of the telescope used and the reasons why the laws of optics would change at all. In the world of science, it is not enough to propose something and await refutation, as Popper imagined–it is also necessary to make *an argument* for the validity of the statement. The structure of a scientific statement looks like the following:

$A$ is valid because of reason $R_1$, because of reason $R_2$, because of reason $R_3, \ldots$

The longer the list of arguments, the better the scientific quality of the statement and its credibility will be. In this expression, the comma replaces a new associative and commutative connective–given that the order or grouping of the arguments does not change the result–as explored in this chapter. If we use the symbol $\oplus$ to refer to this new

[110] This statement was essentially formulated by a Jewish fundamentalist in his attempt to deny the age of the Earth. He would add that if no one had ever seen a dinosaur, then no one could be sure they ever existed. The inconvenience with this argument is that it can also apply to atoms, quarks, Jupiter's storms and many other more cases usually studied by science.

[111] In his novel *Jurassic Park* (1990), Michael Crichton (1942, 2008) proposes a refutation of the first statement through the "fabrication" of a dinosaur by reconstructing its DNA through fossils preserved in resin. The second statement is potentially refutable by means of an artificial probe and experimental measurements.

connective, we have the following expression:

$$A = R_1 \oplus R_2 \oplus R_3 \oplus \cdots$$

$R_1, R_2, R_3, \ldots$ become increasingly "independent" and "complementary" the statement's credibility will increase as well. Below are some classic examples:

- When Newton formulated his theory of universal gravitation, he presented few arguments: how bodies fell to the ground in Earth, the motion of the Moon, the planetary system and the comets. All of these arguments, independent among themselves, supported the idea of universal gravitation.
- When Darwin formulated natural selection, he argued through independent cases: the selection of animals and vegetables by human cultivation, the flora and fauna of the Galapagos and many other specific cases.
- When Einstein formulated General Relativity, he only had one experimental proof: Mercury's perihelion shift. Observation of the 1919 eclipse added the deviation of the light upon passing near the Sun's mass. With the years, other phenomena have been included and yet others will be incorporated until the theory is (eventually) refuted.

This is a common work method in science. However, it poses major difficulties when dealing with binary logic. This method *cannot be formalized in binary logic*. In fact, there is no logical connective allowing us to link a chain of statements so that their truthfulness is *reinforced*. Furthermore, in binary logic, a statement can only be true or false—it can never be more or less credible.

It seems clear that the process of refuting a statement is also an argumentative process. Refutation consists in accumulating arguments to oppose it, just as when defending a thesis. This process can only be comprehended and formalized through dialectic logic. In the natural sciences, for instance, the formal model from mathematics is not followed because the situation is different.[112] Both truth and refutation imply a process of accumulating arguments, instead of a search for a formal counterexample. At this point, dialectic logic, with its multiple logical values, allows us to shed light on the situation.

For Popper, this is not a problem, since his idea of argumentation takes place in the realm of binary logic. As in mathematics, refuting implies finding a *counterexample*. There is no such thing as argumentation: only refutation exists, since its logic only allows for something to be true or false.

## Argumentation in a court of law

The clearest example of argumentation occurs in a court of law, where defendant and plaintiff face each other to defend opposing positions. To do this, they resort to a series of arguments that are considered valid–the "evidence" supplied by each litigating party. On occasion, they might give a different interpretation for common arguments. After the argumentation, a jury or judge will *decide which party is more right than the other*, because one set of arguments will convince them of its "truthfulness" by accumulation in quantity. The litigation is then resolved.

The formulation of a thesis and its refutation is a logical mechanism identical to the development of a *judicial process*. Although our everyday lives are full of this kind of activities, binary logic is unable to build a model for the things that take place in court. While the words "true" and "false" are thrown back and forth, they never have the meaning endowed to them within the realm of binary logic. This is something that can only be formalized by dialectics.

In court, a list of facts $H_1, H_2, H_3, \dots$ is presented; these facts are considered to be true. They are so in an ordinary or day-to-day sense, but not in the mathematical or absolute sense. They are what in this study we refer to as a thesis, something in between "true" and "false".

[112] Imre Lakatos (1922, 1974) and other authors defend Popper's argumentative nature in mathematics, with one difference. The argumentation consists in contrasting rival theories, which also applies to mathematics. His thesis is supported by an excellent collection of historical examples.

Ideally, the various arguments must be simple and independent from each other–they cannot be further "decomposed" or "separated" into more basic arguments. This is necessary for their proper use in argumentation.

The set comprised by all the facts is *contradictory in itself*. All of the $H_i$ cannot be true at the same time, because if they were, there would be no reason to discuss or argue. Each litigating party classifies their arguments into three sets: arguments considered "valid" or "significant", those considered "neutral", "indifferent" or "irrelevant", and those considered "non-valid", "insignificant" or "accidental". All in all, we can classify $H_i$ as $A_1, A_2, \cdots A_p$ to refer to facts argued by the prosecutor, $D_1, D_2, \cdots D_r$ a to refer to facts argued by the defense, and $I_1, I_2, \cdots I_s$ to refer to indifferent facts.[113]

The argumentation consists in proving that the facts at hand "are more truthful" or "bear greater weight" than those of the opposite party. This is where the *argumentation function* comes in. Schematically, we have:

$$A = A_1 \oplus A_2 \oplus \cdots A_p \oplus I_1 \oplus I_2 \oplus \cdots I_s$$

$$D = D_1 \oplus D_2 \oplus \cdots D_r \oplus I_1 \oplus I_2 \oplus \cdots I_s$$

where $A$ is the "value" or "weight" of the accusation and $D$, that of the defense. The task of the judge or jury consists in resolving the relation $A \lesseqgtr D$.The greater value of the function will correspond to whoever is "right". The question may eventually remain undecided.

Having reached this point, we simply need to define the argumentation function.

## The argumentation function

The main property in the argumentation function is a type of monotony allowing to "reinforce" the argumentation–obtain a "greater level of truth". If we resort to the interpretation of dialectics, considering the

[113] This is a simplification since indifferent arguments cannot be the same for the two litigating parties.

case that involves only two variables, we need to look for a function $f(x, y)$ with the following property:

$$f(x, y) \geq x \qquad f(x, y) \geq y.$$

The argumentation is thus "reinforced" by the added arguments. It follows that the function is commutative and associative–in no other way could we obtain a succession of arguments without minding the order in which these are presented. It also follows that $f(x, x) = x$ because the reiteration of the same argument adds nothing to its value as truth.[114]

The argumentation displays a peculiar behavior with regards to the values "true" and "false". It is clear that introducing an argument that is absolutely true does not change the value of the argumentation at all; then, $x \oplus 1 = x$ must be met. Conversely, adding a false argument invalidates the entire argumentation, then $x \oplus 0 = 0$. Taking everything into account, the argumentation is defined as follows.

**Definition 31** *An* argumentation function *between two elements* $x, y$ *in a dialectic lattice* **L** *is referred to as a function* $\oplus$ *that complies with the following:*

1. *Is idempotent (I), associative (A) and commutative (C).*
2. *Is rotationally invariant (RI):* $R(x \oplus y) = Rx \oplus Ry$.
3. $x \oplus y \geq x$ *and* $x \oplus y \geq y$.
4. $x \oplus 1 = x$ *and* $x \oplus 0 = 0$.

It follows from Theorem 42, that $\oplus$ is a sum operation in a semi-lattice. The truth table for $\oplus$ is obtained in a very simple manner: by

[114] In daily life, on the contrary, the reiteration of the same argument can create a false increase in its logical value. Joseph Goebbels (1897, 1945), the sinister minister of propaganda of the Third Reich, is credited with: *A lie told once remains a lie but a lie told a thousand times becomes the truth*. While the quote may be false, it conveys the idea well.

exchanging the values 0 and 1 in the lattice and constructing the resulting **OR** function. Figure 25 illustrates this procedure for lattice **2D4**, but it operates in general for lattices of other ranks and numbers. This function is constructed in a manner similar to how the penetration functions are built, but their diagram clearly differs from those in Figures 15 and 20.[115] From this property, we can obtain the truth table presented in Table 21.

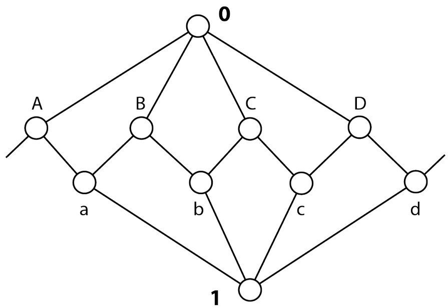


Figure 25: Generic semi-lattice for the argumentative function.

Table 21: Truth table for the argumentative function.

| $\oplus$ | 0 | a | b | c | d | A | B | C | D | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | A | 0 | D | A | 0 | 0 | D | a |
| b | 0 | A | b | B | O | A | B | 0 | 0 | b |
| c | 0 | 0 | B | c | C | 0 | B | C | 0 | c |
| d | 0 | D | 0 | C | d | 0 | 0 | C | D | d |
| A | 0 | A | A | 0 | 0 | A | 0 | 0 | 0 | A |
| B | 0 | 0 | B | B | O | 0 | B | 0 | 0 | B |
| C | 0 | 0 | 0 | C | C | 0 | 0 | C | 0 | C |
| D | 0 | D | 0 | 0 | D | 0 | 0 | 0 | D | D |
| 1 | 0 | a | b | c | d | A | B | C | D | 1 |

Given that it is an **OR** operation with rotational symmetry, this function is commutative, associative, idempotent and rotationally invariant. It is called *argumentative* because, due to the composition

[115] As is evident, in rank-1 lattices, $\oplus$ matches the known **AND** function. In the **3Dn**, lattice, for example, $a \oplus b = p$ but $a * b = 0$.

of dialectic values, an absolute truth–the value 1–cannot be obtained. This function describes the argumentation process in a trial. Each attorney will try to avoid falling into a contradiction–avoid accumulating arguments that lead to 0 value–,which would be fatal for the defense. For this reason, an attorney will never attempt to challenge something that has already been proven as true.

In an actual controversy, depending on the existing number of independent arguments, lattice **rDn** may operate. As the arguments grow in number, $n$ increases, and it is of interest that $r$ also does. Presumably, whoever achieves a combined argument whose logical value is greater than that of their opponent will succeed in court.

This mechanism applies equally when having to choose between two different scientific theories regarding the same phenomena, something which is not very different from how argumentation occurs in a court of law.

## The foundation of principles in science

It is clear that refutation is an argumentation process. But this is not its only application. There are other aspects to how science is constructed, which also make use of argumentation: the construction (or foundation) of scientific principles.

Let us consider some "principles" in the natural sciences:

- Galileo's principle of relativity.
- Galileo's law of inertia.
- Newton's universal gravitation.
- Lavoisier's law of conservation of mass.
- Mayer-Joule's law of conservation of energy.
- Darwin's inheritance and mutation of living beings.
- Darwin's survival of the fittest.

All of these statements are the result of argumentation on simple and isolated observations. On the whole, they are similar to the argumentations presented in court. It is worth noting that the examples taken from physics, chemistry and biology that are shown here have been chosen to show how the method can be applied to all of these cases.

We will now go through the argumentation processes as performed by the different authors. Galileo's principle of relativity was a theoretical formulation resulting from the composition of movements. At the same time, the composition of movements allowed proving that projectiles followed a parabolic trajectory. These were the main arguments. The refutation of Aristotle's physics in relation to falling bodies in a moving vessel followed–a ball dropped from the mast of a moving ship or an arrow thrown towards the stern. Pierre Gassendi (1592, 1655) successfully carried out this experience.[116] The principle of inertia was Galileo's experimental result as part of his analysis of falling bodies: a small ball in a horizontal plane would move with constant velocity.

As we have already proven, see page 31, Newton's argument for gravitation was based on five independent arguments[117]

$$GN = K1_{circular} \oplus K3_{SolarSystem} \oplus K3_{Jupiter} \oplus K3_{Saturn} \oplus K3_{Earth}$$

where $GN$ is Newton's gravitation, $K1$ is Kepler's first law–equal areas in equal time–and $K3$ is the third law–periods are proportional to the 3/2 power of the diameter. The sub-indexes correspond to uniform circular motion, the solar system and the satellites of Jupiter, Saturn and the Earth.

Once the law of gravitation was established, Newton proved that the motion may be elliptical, Kepler's second law, see [67, 68, III, *Theorema* xiii] or parabolic [68, III, *Theorema* xx] and analyzed the trajectory of the 1680 comet discovered by John Flamsteed, the first Royal Astronomer. In [67, 68, III, *Theorema* xix] he proposes a gravitational theory corresponding to the tides.

[116] The difficulty in noticing the Earth's rotation by means of falling bodies was dependent on the law of composition of movements–it was not an objection to the existence of rotation.

[117] In the first edition there were four, $K3$ for Saturn's satellites was missing. His reasoning was a typical "induction" reinforced by the presence of the new argument.

These results can be stated as follows:

$$GN \Rightarrow K2 \qquad GN \Rightarrow \text{Flamsteed comet}$$

where $\Rightarrow$ is the logical implication, $K2$ is Kepler's second law–the motion is elliptical with the Sun being one of the foci–to which he adds the comet's parabolic motion.

As important as "explaining" Kepler's first three laws is having dismissed and ignored the fourth law that the orbits of the six planets correspond to the five regular solids–the octahedron, the icosahedron, the dodecahedron, the tetrahedron, the cube. Why did Newton ignore $K4$? For a simple reason: the theory of gravitation proved that the position of a body's satellites can be located at any distance from the center of attraction.[118]

The principle of mass conservation in chemical reactions was an experimental result obtained by Antoine-Laurent de Lavoisier (1743, 1794), see [54], by means of several experiences in which he weighed the components and composites obtained in various experiments. Each case added a new argument to the formulation of the principle. The analysis of gases, especially oxygen, contributed to refuting the opposite notion–the *phlogiston* theory–which argues that mass is not preserved in the formation of oxides.

In 1842, Julius von Mayer (1814, 1878) posited that oxygen was a major component in the metabolism of living beings and their source of energy. In 1843, James Joule (1818, 1889) established the equivalence between mechanical work and the heat produced by the viscosity of water. This equivalence could also be proven by heating through electric resistance, compression of a gas or heating upon boring a cannon barrel. These independent experiments–and many more performed

[118] The problem of the geometry of the solar system goes as far back as Pythagoras, who associated the orbits to the musical scale. Kepler returned to this subject and the music of the heavens. In 1766, Johann Bode (1747, 1826)—together with his student, Titus–proposed an equation for the distribution of the planets, $d = 0.4 + 0.3 \times 2^n$ where $d$ is measured in astronomical units–the average distance between the Earth and the Sun—and $n = -\infty, 0, 1, 2, \cdots$. While asteroids fulfill this equation, Neptune does not. The issue of the planetary distribution remains open and is now expanded by the discovery of exoplanets.

by other scientists–used the principle of conservation of energy as an argument.

Darwin's argumentation appears in *The origin of Species* [15]. It is hard to find a clearer and more complete example of argumentation in science. Each chapter of the work incorporates arguments that reinforce his basic formulations. Chapters I and II show natural variations of the species and those achieved by domestication. In chapter III–using Malthus' argument on the free growth of populations–he proves that not every living being that is born, survives. In chapter IV, he theorizes about the natural selection of the survivors. In chapter V, he establishes different manners of adaptation. In chapters X and XI, he analyzes the problem from a geological standpoint and shows its plausibility. In chapters XII and XIII, he studies the geographical distribution of living beings.

Darwin's work complies with the theory of argumentation with precision. Not only does it contain an extensive list of simple arguments, independent among themselves, but opposing arguments are also present. In chapters VI and VII, he is brave enough to analyze the flaws in his argumentation. The book in its entirety affirms the principles through the application of the argumentation function. Chapter XV summarizes the complete argumentation.

# Implication

## Introduction to dialectic implication

The traditional use of dialectics does not need an implication function. Implication allows us to build fixed and unchangeable chains of reasoning. Conversely, dialectic thinking preoccupied itself with movement, change, and has no need for this function. However, human logic, taken as a whole, must contemplate both aspects of thought. This is the first reason for analyzing the implication function within a dialectic lattice.

The second reason is linked to the foundations of implication. In binary logic, things are very simple and the properties of implication are connected amongst themselves. In dialectics, implication functions show us a very different scenario and many results which are accepted as valid and "proven" in binary logic, have counterexamples that refute these demonstrations.

The main purpose of the formalization of this logic was to describe the shape of "proper reasoning". From Aristotle to Boole, to Gottfried Wilhelm Leibniz (1646, 1716), the issue was under discussion. Towards the end of the $19^{th}$ century and the beginning of the $20^{th}$, it appeared to have been settled. Frege and Russell's formalization materialized a well-rounded, flawless construction.[119]

Binary logic more than succeeded in describing "logical reasoning" by introducing the binary function of implication $\Rightarrow$ and its basic properties. This function is one of the most important in binary logic, comparable to **AND** and **OR**, and is equivalent to these to a large extent, as it is not difficult to prove.

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[119] Regardless of this alleged perfection, some scientists were wary of the results. Henri Poincaré doubted the correctness and definitive nature of binary logic. It is possible that he sensed that human thought had more structure than what binary logic could describe.

Dialectic logic is an extension of binary logic that includes other forms of thought which escape the binary model. Since implication is concerned with the construction of formal theories, it does not occupy an important place in dialectics. Nevertheless, given that dialectics must contain binary logic as a special case, extending the definition of implication becomes necessary. However, this is not expected to add new forms of construction of deductive knowledge.

The final stage of consolidation of scientific knowledge is the formulation of a deductive theory. Axioms or principles or hypotheses take part in the construction of a theory, which serve as accepted propositions with a certain logical level. From these basic, accepted propositions, new statements are constructed by means of the application of a reduced set of formal structures taken as valid.

In dialectic logic, the problem of extending the definition of implication is quite complex. It comprises two sets of rules:

- *Formal rules* in order to construct new statements from statements accepted as valid.
- *Semantic rules* that allow us to apply dialectics to cases that are of interest to science or history.

Ultimately, defining the rules of reasoning used in mathematics and the sciences is what matters. Creating a priori functions, as binary logic has done, proves to be useless. Frederic Fitch (1908, 1987), see [25, 91]. has proposed a different path. In essence, he defined a set of rules that allow building a valid argument. In the upcoming sections related to the implication function we will use this formalization, along with the necessary adaptations.

## The formal rules of dialectic implication

This section will cover these formal structures and how they extend to dialectic logic. Disregarding obvious repetitions,[120] these structures

[120] The equivalence *–co–implication* in Fitch– of propositions is something redundant which does not contribute anything, which is why it has been omitted from the formal rules.

appear below. Each rule has a mnemotechnic acronym to simplify its use throughout the exposition.

1. Conjunction introduction (CI): if $a$ and $b$ are theses, $a \,.\, b$ is one as well.[121]

2. Disjunction introduction (DI): $a \Rightarrow a + b$ is a thesis, regardless of $b$.

3. Conjunction elimination (CE): if $a \,.\, b$ is a thesis, $a$ and $b$ are theses as well.

4. Disjunction elimination (DE): if $a + b$ is a thesis and $a \Rightarrow c$ and $b \Rightarrow c$ are theses then $c$ is a thesis. This rule allows separating a demonstration into two (or more) simples cases.

5. Principle of double negation (PDN): if $a$ is a thesis, $NN\, a$ is one as well and reciprocally.[122]

6. Transitive property of implication (T): if $a \Rightarrow b$ and $b \Rightarrow c$ are theses, then $a \Rightarrow c$ is one as well. This rule allows us to form chains of demonstration.

7. *Modus Ponens* (MP): if $a \Rightarrow b$ and $a$ are theses, then b is also a thesis.[123] This rule allows us to cut chains of demonstration.

8. *Modus Tollens*, in its simple variant (MT): if $a \Rightarrow b$ is a thesis, then if $b$ is 0, $a$ is 0. This rule implies that it is possible for $0 \Rightarrow 0$

[121] This rule is accepted in an acritical manner. However, there is reason to believe that it is not generally valid. Quantum mechanics has shown this time and again. Let us consider these two statements: $a$ the position of a particle in a given instant can be precisely measured, and $b$ the speed of a particle in a given instant can be precisely measured. While both statements are correct, $a \,.\, b$ is false and this is referred to as Heisenberg's uncertainty principle. Quantum mechanics has challenged logic, as we have shown before. This observation has major consequences for dialectic implication.

[122] In some natural languages–Spanish is one example–, double negation can have *an emphatic purpose*, as opposed to one of affirmation.

[123] *Modus Ponens* can be stated in a less aggressive manner to binary logic, while remaining equivalent. If the first two propositions are not false, the third one is not, either.

to be a thesis. *Modus Tollens* extended (MTE) is a stricter condition, if $a \Rightarrow b$ is a thesis, then $N\,b \Rightarrow N\,a$ is also a thesis.

9. Principle of contradiction (PC) and the principle of contradiction, extended (PCE): if $a \Rightarrow N\,a$ is a thesis, then $N\,a$ is also one (PC). The extended principle establishes that if $a \Rightarrow b$ and $a \Rightarrow N\,b$ are theses, then $N\,a$ is one as well. PCE is a consequence of PC and the previous rules.[124] PC is a specific case of PCE.

Before continuing with this subject, we need to clarify an essential point on CI. If we consider two atoms, $a, b$ of the lattice. It is clear that both are theses. However $a \,.\, b = 0$, then, CI is not met. This result is not only general for all dialectic lattices, but CI is not met as well for many other lattice elements, for example, $a$ and $N\,a$ are theses, but $a \,.\, N\,a = 0$ if $N$ is strict. We already know that the CI rule is not generally valid. For this reason, in the exposition that follows, the expression "meets the formal rules" means "meets the formal rules, except for CI". When this is an important issue, it will be explicitly indicated. This topic is analyzed in detail in what follows and is an essential issue for the theory of dialectic implication.

## The semantic rules of dialectic implication

Semantic rules are additional conditions asked from an implication function for it to yield coherent results from the actual, spontaneous use of dialectics.

1. Principle of permanence (PP): the values from binary logic must be respected, then $0 \Rightarrow 0 = 1$, $0 \Rightarrow 1 = 1$, $1 \Rightarrow 0 = 0$, $1 \Rightarrow 1 = 1$.

2. Rotational invariance (RI): the implication function must be invariant in the rotation of the lattice **rDn** where it is defined, this

[124] The demonstration is the following: If we take as a hypothesis: 1. $a$; 2. $a \Rightarrow b$ and 3. $a \Rightarrow Nb$ it follows that: 4. $b$ due to MP in 1 and 2; 5. $NNb \Rightarrow Na$ due to MTE in 3; 6. $b \Rightarrow Na$ due to PNN in 5.; $Na$ due to MP in 4 and 6. Then, $Na$ is a thesis.

is $R(x \Rightarrow y) = Rx \Rightarrow Ry$. As a result, if $x \Rightarrow y$ is a thesis, so is $Rx \Rightarrow Ry$.

3. Idempotency (I): For every dialectic value $d$, $d \Rightarrow d = d$ is met.
4. Principle of mixture (PM): In some applications, it is desirable for dialectic values to avoid "mixing" when repeatedly using the implication function. With more precision, there is a non-trivial subset of lattice elements, $S$ such that if $x, y \in S$ then $x \Rightarrow y \in S$.[125]

The application of the PM rule applies especially to the analysis of the natural sciences, as analyzed further ahead. When we say that "it meets the semantic rules", we mean to say that "it meets the semantic rules, except for PM". The importance of these additional rules–which are not necessary in binary logic–will be clear from the exposition in this chapter and those that follow.

## Non-contradiction and independence of the rules

The non-contradiction of the formal rules follows, since in binary logic these are met, therefore the set of rules is not contradictory. The non-contradiction of the semantic rules, amongst themselves and with the formal rules, results from the existence of implication functions that meet them. Therefore, in lattice **D3** for instance, this set of rules allows finding 16 two-variable functions that meet them, except, naturally, for CI and eventually, PM.[126]

Independence of the rules–something that does not bear much interest in practice–implies finding counterexamples of structures that comply with all the rules except for those we wish to investigate.[127] There are several possible cases for rules of one type or the other:

[125] In every lattice, $S = (0, 1)$ is a valid example, it is binary logic. In **Dn** $S = (0, a, 1)$ or in **2Dn**, $S = (0, a, b, A, 1)$ are also valid examples.

[126] This number was established by a software program which systematically explored every possible case. In lattice **D3** a two-variable truth table has 5×5=25 values to be determined. Each one of these may adopt 5 values, then there are $5^{25} \approx 2.98 \times 10^{17}$ possible two-variable functions. Of course, there are much less implication functions defined by the rules.

[127] This way of proving independence comes from the famous work by David Hilbert (1862, 1943), *Grundlagen der Geometrie* (1899) [46] (Foundations of Geometry) where

- The rule is met in all the dialectic lattices. For example: PDN and CE are met in every dialectic lattice.[128]
- The rule is not met in any dialectic lattice. This is the case with CI, as already mentioned.
- The rule can be derived from other rules, it is not independent.
- The rule is independent from a certain set of other rules.
- It makes no sense to consider this rule. This is the case of PP, which establishes the coherence with binary logic.

We will begin by analyzing the semantic properties. Failing to comply with the PP rule means violating some of the formal conditions. If $0 \Rightarrow 0 = 0$ MTE is not met given that $N0 \Rightarrow N0 = 1$ and it is not 0.[129] If $1 \Rightarrow 0 = 1$ MP is not met, given that a thesis implies a false value. If $0 \Rightarrow 1 = 1$ MT is not met since in a thesis statement, if the consequent is a thesis, the antecedent is one as well. If $1 \Rightarrow 1 = 0$ DE is not met given that $a \Rightarrow 1$ and $b \Rightarrow 1$ are theses but $1 = a_b \Rightarrow 1$ is not. None of these cases can have a dialectic value since this would contradict RI. To sum up, the PP rule is a consequence of rules MP, MT, MTE, DE and RI. Therefore, it can be omitted from the set of conditions.

The truth table in Table 22 does not meet RI–because $0 \Rightarrow a = a$ is a thesis but $0 \Rightarrow b = a$ and not $b$–but it does all the others. This counterexample–out of many possible ones–shows the independence of this property from the remaining ones.

The truth table in Table 23 does not meet I–because $a \Rightarrow a = 0$ instead of $a$ but it does the remaining properties, then I is independent.

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he systematically constructed "geometries" to demonstrate the independence of his many axioms. Undoubtedly, Hilbert has been the father of formalism due to his works and his proposals for topics of research.

[128] If $a$ is a thesis, $NNa$ is also one, regardless of whether $a = 1$ –which is invariant– or if it is a dialectic value, since the automorphism $NN$ transforms a dialectic value into another. The reciprocal case is proven in a similar manner. EC is also met because if $a \,.\, b$ is a thesis, it is clear that neither $a$ nor $b$ are 0, then, they are theses.

[129] It is worth asking, in this line of reasoning—and those that follow—what logic are we applying? The answer is immediate: binary logic. For this, the reasoning must be written without misusing the language. *Hypothesis* $0 \Rightarrow 0 = 0$ *with logical value true. MTE is applied and then* $N0 \Rightarrow N0 = 1 \Rightarrow 1 = 1$ *also has logical value true and is not* 0. *Then, this is the case of the PCE rule of binary logic. Then the hypothesis is false.*

Table 22: Implication function in **D3** which does not meet RI.

| ⇒ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | a | a | 1 |
| a | 0 | a | 0 | 0 | a |
| b | 0 | 0 | b | 0 | a |
| c | 0 | 0 | 0 | c | a |
| 1 | 0 | 0 | 0 | 0 | 1 |

Table 23: Implication function in **D3** which does not meet I.

| ⇒ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | b | c | 1 |
| a | 0 | 0 | 0 | 0 | a |
| b | 0 | 0 | 0 | 0 | b |
| c | 0 | 0 | 0 | 0 | c |
| 1 | 0 | 0 | 0 | 0 | 1 |

Table 24 presents an example–out of the 12 possible cases–of an implication that does not meet PM–with a "mixture" of dialectic values–that meets the remaining conditions.

With this, the non-contradiction and independence of the semantic properties is proven. The case of formal properties is slightly more complicated, as illustrated by Figure 26. This figure–which is, of course, not to scale–presents the relations of dependence for the different properties. The external rectangle indicates that there are 15,625 functions in **D3** that meet the properties RI, I and also, as in every dialectic lattice, CE and PDN. The other rectangles indicate the properties and the number of cases associated with each property until arriving at the 16 implication functions that meet all the properties.

Table 24: Implication function in **D3** which does not meet PM.

| ⇒ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | b | c | 1 |
| a | 0 | a | 0 | 0 | b |
| b | 0 | 0 | b | 0 | c |
| c | 0 | 0 | 0 | c | a |
| 1 | 0 | 0 | 0 | 0 | 1 |

Naturally, the independence of some properties is deduced from this diagram. Thus, for example, T is independent from MP, MT, PCE, given that possible counterexamples exist. The same thing happens with MTE. MP and MT are independent from PCE due to the existence of counterexamples. PCE is also independent from DE, DI is independent from DE.

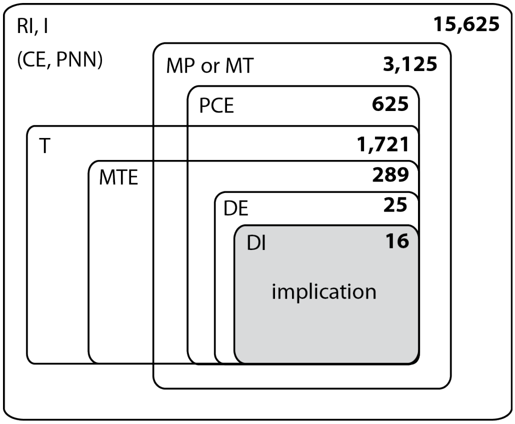


Figure 26: Implication in **D3** and the formal properties.

We will begin by analyzing Russel's classical definition of the implication function $x \Rightarrow y = N\,x + y$ which can be extended directly to dialectic lattices under the simple condition that $N$ be a strict negation. Table 25 presents the truth table for **D3**. This is of interest since it serves as a counterexample for many formal properties.

Table 25: Truth table of the classic implication in **D3**.

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | b | 1 | b | 1 | 1 |
| b | c | 1 | 1 | c | 1 |
| c | a | a | 1 | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

This implication function does not meet MT or MP–for instance, $a \Rightarrow 0 = b$ is a thesis–but it does meet MTE. It does not comply with

T, for instance, because $1 \Rightarrow a = a$ and $a \Rightarrow 0 = b$ are theses and lead to $1 \Rightarrow 0$ being a thesis. It does not meet DE, because $a \Rightarrow 0 = b$ and $b \Rightarrow 0 = c$ are theses but $1 = a + b \Rightarrow 0$ is not. It fails to comply with PCE, because for $1 \Rightarrow a = a$ and $1 \Rightarrow Na = b$ they are theses, but $N1 = 0$ is not. On the other hand, it complies with PDN, DI and CE.

We will consider several cases for purposes of exemplifying the dependence of the properties. The DI rule is independent because there exists a counterexample in **D3** and it is the function given by the truth table in Table 26. In fact, $a \Rightarrow a$ is a thesis, but $a \Rightarrow a + b$, that is, $a \Rightarrow 1$ is not. The remaining rules are met, except for, naturally, CI.

Table 26: Implication function in **D3** which does not meet DI.

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 1 |
| a | 0 | a | 0 | 0 | 0 |
| b | 0 | 0 | b | 0 | 0 |
| c | 0 | 0 | 0 | c | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 |

The DE rule has counterexamples in **D3**, one of which is presented in Table 27. In fact, $a + b = 1$ is a thesis. $a \Rightarrow a$ and $b \Rightarrow a$ are theses, but $a + b \Rightarrow a$ is not a thesis. All the other rules are met, except for, naturally, CI.

Table 27: Implication function in **D3** which does not meet DE.

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | b | c | 1 |
| a | 0 | a | c | b | a |
| b | 0 | c | b | a | b |
| c | 0 | b | a | c | c |
| 1 | 0 | 0 | 0 | 0 | 1 |

The MTE rule has counterexamples in **D3**, one of which is presented in Table 28. In fact, $a \Rightarrow 1$ es una tesis, pero $0 \Rightarrow Na = b$ is not. It fails to meet DI as well, because $0 \Rightarrow 0$ is a thesis but $0 \Rightarrow a = 0 + a$ is not. All the other rules, *including* MT, are met, except for, naturally, CI.

Table 28: Implication in **D3** which does not meet MTE or DI.

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 1 |
| a | 0 | a | 0 | 0 | a |
| b | 0 | 0 | b | 0 | b |
| c | 0 | 0 | 0 | c | c |
| 1 | 0 | 0 | 0 | 0 | 1 |

This example reveals an error in the classical deductive systems. In [25, II, 10.23] we can find the following formal scheme (except for the change in notation) which "proves" MTE:

| | | |
|---|---|---|
| 1) | $p \Rightarrow q$ | hypothesis |
| 2) | $p + Np$ | hypothesis |
| 3) | $Nq$ | subordinate hypothesis |
| 4) | $p + Np$ | reiteration of 2), subordinated |
| 5) | $p$ | new hypothesis in second subordination |
| 6) | $p \Rightarrow q$ | reiteration of 1), in second subordination |
| 7) | $q$ | MP between 5) and 6) |
| 8) | $Nq$ | reiteration of 3), in second subordination |
| 9) | $Np$ | PCE from 5), 7) and 8), in the subordinate |
| 10) | $Np$ | DE of 4) due to 5) to 9) |
| 11) | $Nq \Rightarrow Np$ | implication from 3) to 10). |

This reasoning only uses the formal properties MP, PCE and DE, as complied with in the truth table of Table 28, complies with, which we know does not meet the property MTE and is a counterexample.[130] The error in reasoning–which it is not hard to fall into–is found in the incorrect application of DE in line 10. A statement can be reiterated in a subordinate, but not the other way round. Statement 9 cannot be applied to line 4. A similar error can be found in [25, II, 10.22].

If we take the case of **D3**, we also obtain:

- Rule T is met in all the functions with MTE and DE.

[130] There are other cases in **D3** which are also counterexamples, such as the case of $f_1, f_2, f_3$–see further below–respectively $0, x, 0$ or $0, 1, 0$.

- Rule MP is met in all the functions with PCE.
- Rule PCE is met in all the functions with MTE and DE.

These examples are enough to give ud an idea of the complexity of the subject of independence of the formal rules. Due to the speed of calculation, it is not simple to analyze more complex dialectic lattices given that the number of truth tables to examine in **rDn** grows by $m^{m^2}$ where $m = r\,.\,n{+}2$ is the number of elements to consider in the lattice.

For the simplest case of a two-tier dialectic lattice, **2D4**, we have that $m = 9$ and, in consequence, $9^{81} \approx 1,97 \times 10^{77}$ cases to be examined. Since this matter does not bear much practical interest, we will not delve into this issue.

## Implication functions in general

In this section, we will analyze the issue of implication functions in any given dialectic lattice. The starting point is Table 6, page 115, which presents the truth table for an RI-compliant function. The properties that make it specific to the case at hand must be added to this.

According to Theorem 30, for the implication function to be rotationally invariant, the five functions in the truth table must be so as well. We will analyze the case of $f_1(y), f_2(x), f_3(y)$ and $f_4(x)$. The only functions $f_1(y)$ that are invariant with regards to the automorphisms of the dialectic lattice elements are $0, y, R\,y, RR\,y, \ldots, 1$, where $R$ is the rotation of the elements and comparable functions for the remaining cases. The function $g(x, y)$ must be analyzed in each case.

**Theorem 53** *For the formal and semantic properties of implication to be met,* $f_1 > 0$*,* $f_2 > 0$*,* $f_3 = 0$ *and* $f_4 = 0$ *must occur.*

**Proof**. The MP property requires that the function $f_4 = 0$. In fact, if $d \Rightarrow 0$, is a thesis, where $d$ is a dialectic value, then 0 would also a thesis. This occurs for all dialectic values.

If $f_3 \neq 0$, for example, with $1 \Rightarrow a$ being a thesis, then MTE calls for $Na \Rightarrow 0$ to be one as well, against MP.

The DE property requires that expressions such as $B \Rightarrow A$, where $A, B$ are maximum elements, be false. In fact, if $A \Rightarrow A$ and $B \Rightarrow A$ are theses, then it is concluded that $A + B \Rightarrow A$, that is, $1 \Rightarrow A$ and due to MTE, $N\,A \Rightarrow 0$ against $f_4 = 0$.

If $f_1 > 0$ then $0 \Rightarrow x$ is a thesis, then, due to MTE, $Nx \Rightarrow 1$ is also a thesis and then, $f_2 > 0$. If $f_2 > 0$, then it also occurs that $f_1 > 0$. Let us consider two maximum elements $A, B$. From thesis $A \Rightarrow A$ we obtain, due to IC, that $A \Rightarrow A + B = 1$, is also a thesis, it is then proven.

If $f_2 = 0$, then we will consider two maximum elements, $A, B$, It is clear that $A$ is a thesis, but $A \Rightarrow A + B = 1$ is not, then, CI would not be met. Then $f_2 > 0$ and also $f_1 > 0$ because if it were not, then $f_2$ would not be either. □

Table 29 presents the general structure of the implication function $x \Rightarrow y$ as already proven. Since this is obvious, we only need to determine two functions of a variable and one of two variables, which must comply with the formal properties of implication.

Table 29: Truth table of the generic implication in **rDn**.

| $\Rightarrow$ | 0 | dialectic | 1 |
|---|---|---|---|
| 0 | 1 | $f_1(y)$ | 1 |
| dialectic | ⋯ 0 ⋯ | $g(x, y)$ | $f_2(x)$ |
| 1 | 0 | $\cdots 0 \cdots$ | 1 |

From the previous properties of functions $f_1, f_2$ we have that the only valid cases in **rDn** –without considering rotations, the PM case– are the four presented in Table 30. Rotations increase the number of possible functions.

The RI rule–the function is invariant in one rotation–has already been used in other functions of dialectic logic. There is nothing special about it. The analysis of the function $g(x, y)$ is somewhat more complex and relates to formal and semantic rules. The I property requires that $g(x, x) = x$ for all values.

In order to determine the potential implication functions, it is enough to verify the remaining formal properties. The systematic search for

Table 30: Auxiliary functions in **rDn**.

| | $f_1$ | $f_2$ |
|---|---|---|
| 1 | $y$ | $x$ |
| 2 | 1 | $x$ |
| 3 | $y$ | 1 |
| 4 | 1 | 1 |

functions only yields 16 cases in the **D3** lattice. Table 31 shows the 4 implication functions without "mixing".

Table 31: The 4 implications without "mixing" in **Dn**.

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | b | c | 1 |
| a | 0 | a | 0 | 0 | a |
| b | 0 | 0 | b | 0 | b |
| c | 0 | 0 | 0 | c | c |
| 1 | 0 | 0 | 0 | 0 | 1 |

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | 0 | a | 0 | 0 | a |
| b | 0 | 0 | b | 0 | b |
| c | 0 | 0 | 0 | c | c |
| 1 | 0 | 0 | 0 | 0 | 1 |

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | a | b | c | 1 |
| a | 0 | a | 0 | 0 | 1 |
| b | 0 | 0 | b | 0 | 1 |
| c | 0 | 0 | 0 | c | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 |

| $\Rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | 0 | a | 0 | 0 | 1 |
| b | 0 | 0 | b | 0 | 1 |
| c | 0 | 0 | 0 | c | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 |

The remaining cases are completed by replacing $x$ for $Rx, RRx, \cdots$ –and similarly for $y$– in a **Dn** lattice.

## Basic implications in **rDn**

It is possible to find implication functions in every dialectic lattice **rDn**. There is a natural way of constructing implication functions for each lattice–that meet the appropriate semantic and formal properties–by means of the lattice's relation of order (with the exception of $0 \Rightarrow y$ cases).

**Definition 32** *In a dialectic lattice* **L**, *for two elements* $x, y$, minor basic implication functions *are defined, such as: 1) for* $x \neq 0$, *then for* $x \leq y$ *it is defined that* $x \Rightarrow y = x$; *2) for* $x \nleq y$ *it is defined that* $x \Rightarrow y = 0$; *3) for binary values PP is met; 4) the* $f_1$ *functions may be* $y$ *or 1 and* $f_2$ *may be* $x$ *or 1.*

While not in an analytical manner but in its basic idea, this definition extends the classical definition: $0 \Rightarrow 0 = 1$, $0 \Rightarrow 1 = 1$, $1 \Rightarrow 1 = 1$ and $1 \Rightarrow 0 = 0$, that is, $x \leq y$ is true and $x \nleq y$ is false.

**Theorem 54** *The function defined in 32 meets the formal properties DI, CE, DE, PNN, T, MP, MTE, PC and the semantic properties PP, I, RI and PM.*

**Proof**. We will demonstrate each one of the properties in order, first for $x \neq 0$ and then for $x = 0$:

DI If $x \Rightarrow y = x$ then $x \leq y \leq x + y$, then $x \Rightarrow x + y = x$ is a thesis. If $0 \Rightarrow y$ its value is $y$ by definition of $f_1$ and it is a thesis. The case $y = 0$ is met by definition.

CE If $x \,.\, y$ are theses, then $x \,.\, y > 0$ and it is met that $x > 0$ and $y > 0$, Then, both are theses. The case $x = 0$ has no use.

MP If $x, x \Rightarrow y$ are theses, $0 < x$ and $x \leq y$. Then $0 < x \leq y$. Then $y$ is a thesis. The case $x = 0$ has no use.

DE If $x + y$ is a thesis and $x \Rightarrow z$, $y \Rightarrow z$, then $x \leq z$, $y \leq z$, then $x + y \leq z$, that is, $x + y \Rightarrow z$. Due to MP, $z$ is a thesis. The case $x = 0$ is met due to the definition of $f_1$.

PNN If $x$ is a thesis, $NNx$ is one as well, in the case $x = 1$ due to the definition of the negation and in the remaining case, due to the automorphism $NN$ transforming one dialectic value into another. The case $x = 0$ has no use.

T The relation of order in the lattice has the transitive property. Then, if $x \Rightarrow y$ and $y \Rightarrow z$ are theses, it is met that $x \leq y \leq z$

then $x \Rightarrow z$ is a thesis. The case $x = 0$ holds true, due to the definition of $f_1$, $0 \to y = y$ that is, $0 < y \leq z$ and $z$ are theses.

MTE If $x \Rightarrow y$ complies with $x \leq y$, then $Ny \leq Nx$ due to the definition of the negation, where it results in that $Ny \Rightarrow Nx$. If $0 \Rightarrow y$, which is a thesis by the definition of $f_1$, it is met that $Ny \Rightarrow 1$ by the definition of $f_2$. If $x \Rightarrow 1$, which is a thesis by the definition of $f_2$, $0 \Rightarrow Nx$ holds true, which is a thesis by the definition of $f_1$.

PCE If $x \Rightarrow y$ and $x \Rightarrow Ny$ are theses, then if $x = 0$, which may occur due to the definition of $f_1$, it is clear that $N0 = 1$ is a thesis. The case $x = 1$ is impossible since $f_3 = 0$. If $x$ has a dialectic value, then $Nx$ is a thesis.

PP Is part of the definition of the implication function.

RI The function is rotationally invariant because both $f_1$ and $f_2$ are, as well as the relation $x \leq y$.

I It is clear that $d \leq d$, then $d \Rightarrow d = d$ for every dialectic value $d$.

PM The principle of mixture is met in some cases due to $d \Rightarrow d = d$ and the definitions of $f_1$ and $f_2$ are without rotation of the elements.

Finally, since we have two possible alternatives for f1, f2, there are 4 functions defined in this way. The result is general for all **rDn** lattices. With this, the theorem is proven. □

It is important to note that this theorem is valid both for *common as well as exotic negations*. Only the monotony property and the one that states that the negation of a dialectic value is another dialectic value are involved in the demonstration. It is also not necessary for it to be a *strict negation*. A major consequence from a practical point of view consists in noting that, in order to do a systematic search for implication functions, it is enough to use $N_0$, for instance. The function thus obtained is valid for all the other negations of the lattice by RI and the relations between negations and rotations.

**Theorem 55** *The implication function $x \Rightarrow y$ defined is monotonous, decreasing in $x$ and increasing in $y$, for the dialectic values.*

**Proof**. Let us consider only dialectic values. If $z \leq x$, since $x \leq y$ then $z \leq y$, then $z \Rightarrow y$ is a thesis. It is similarly proven for $y \leq w$. □

A consequence of this theorem allows obtaining other implication functions with similar characteristics to those in this theorem.

**Theorem 56** *If in the definition of minor implication, condition 1) is replaced by "for $x \neq 0$, then for $x \leq y$, $x \Rightarrow y = y$ is defined", 4 functions are also obtained that meet the formal properties DI, CE, DE, PDN, T, MP, MTE, PC and the semantic properties PP, RI, I, PM. They are referred to as* major basic implications.

**Proof**. In the previous demonstration, only the value of the implication function appears in property DI, then all the other properties are valid. In the case of DI, only the following text is modified in the demonstration: "If $x \Rightarrow y = y$ then $x \leq y \leq x + y$, then $x \Rightarrow x + y = y$ is a thesis." □

In **Dn** lattices, major and minor implications *match*–this is not the case in **rDn** lattices of rank greater than 1, as can be seen below.

There are other implication functions which are neither minor nor major, for example, the one presented in Table 23: by failing to comply with I, it does not derive from the relation of order that this condition calls for. If the RI condition is not required in **D3**, 4080 implication functions are possible.

The truth table partially contains a diagonal table. Let us assume that we were to order the table values by increasing dialectic values. After 0, atoms appear and so on. Figure 32 illustrates this situation.

## Implications in **Dn**

We will consider the *basic implication* in **D4** as an example of the general case in **Dn**. According to the proposal of the order relation, Table 33 gives us the truth table. This implication allows us to obtain uni-

Table 32: Analysis of implication in **rDn**.

| $\Rightarrow$ | 0 | a | $\cdots$ | k | $\cdots$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | $f_1$ | | | $f_1$ | 1 |
| a | 0 | a | $\cdots$ | 0 | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $f_2$ |
| k | 0 | 0 | $\cdots$ | k | | |
| | 0 | | | | | |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $f_2$ |
| | 0 | | | | | |
| 1 | 0 | 0 | | | 0 | 1 |

versally valid results based on dialectic statements. We will take the statement $a \Rightarrow x = a$ as an example, where $x$ is the unknown. Since $a$ is a thesis, we conclude that the possible values are $x = a, 1$ as a consequence of Modus Ponens.

Table 33: Table of implication in **D4**.

| $\Rightarrow$ | 0 | a | b | c | d | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | a | b | c | d | 1 |
| a | 0 | a | 0 | 0 | 0 | a |
| b | 0 | 0 | b | 0 | 0 | b |
| c | 0 | 0 | 0 | c | 0 | c |
| d | 0 | 0 | 0 | 0 | d | d |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |

Aside from these implication functions, there are the three additional ones with value 1, such as those presented in Table 31 for **D3**.

**Theorem 57** *In* ***Dn****, basic implications are the only possibility.*

**Proof**. Let us in fact assume that, for example, $a \Rightarrow h$ is a thesis, where $a, h$ are two dialectic values. $R$ is the lattice rotation. There is a rotation $R^k$ such that $h = R^k\, a$. We have, then, that if $a \Rightarrow R^k\, a$ is a thesis, it also is one by rotating $R^{n-k}$, then $R^{n-k}\, a \Rightarrow a$ is a thesis, given that $R^n$ in **Dn** is the identity. If $R^{n-k}\, a \neq a$, because of the DE property, from $a \Rightarrow a$ and $R^{n-k}\, a \Rightarrow a$ we obtain $1 = a + R^{n-k}\, a \Rightarrow a$ contradicting that $f_3 = 0$. Then, by applying the rotation, the function $g(x, y)$ can only be different from 0 in the diagonal. Rule I completes the demonstration. □

## Implications in **2Dn**

Table 34 presents the truth table for the minor basic implication function in lattice **2D4** as the first example of the general case.

Table 34: Table of the minor implication in **2D4**.

| ⇒ | 0 | a | b | c | d | A | B | C | D | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | a | b | c | d | A | B | C | D | 1 |
| a | 0 | a | 0 | 0 | 0 | a | 0 | 0 | a | a |
| b | 0 | 0 | b | 0 | 0 | b | b | 0 | 0 | b |
| c | 0 | 0 | 0 | c | 0 | 0 | c | c | 0 | c |
| d | 0 | 0 | 0 | 0 | d | 0 | 0 | d | d | d |
| A | 0 | 0 | 0 | 0 | 0 | A | 0 | 0 | 0 | A |
| B | 0 | 0 | 0 | 0 | 0 | 0 | B | 0 | 0 | B |
| C | 0 | 0 | 0 | 0 | 0 | 0 | 0 | C | 0 | C |
| D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | D | C |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Naturally, functions $f_1, f_2$ from Table 30 also apply, that lead to three additional implication functions. In **2Dn** lattices, the major basic implication is different from the minor, and Table 35 presents an example of a truth table.

As in the minor case, replacing $f_1, f_2$ with the value 1 yields three additional implication functions.

**Theorem 58** *In **2Dn**, basic implications are the only possibilities that comply with the PM condition.*

**Proof.** In order to demonstrate the theorem, we will refer to atoms as $d_i$ and to the maximum elements of **2Dn** as $D_i$. Sub-index $i$ is considered n-module. If $R$ is the lattice rotation and $N$ is the negation

Table 35: Table of the major implication in **2D4**.

| $\Rightarrow$ | 0 | a | b | c | d | A | B | C | D | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | a | b | c | d | A | B | C | D | 1 |
| a | 0 | a | 0 | 0 | 0 | A | 0 | 0 | D | a |
| b | 0 | 0 | b | 0 | 0 | A | B | 0 | 0 | b |
| c | 0 | 0 | 0 | c | 0 | 0 | B | C | 0 | c |
| d | 0 | 0 | 0 | 0 | d | 0 | 0 | C | D | d |
| A | 0 | 0 | 0 | 0 | 0 | A | 0 | 0 | 0 | A |
| B | 0 | 0 | 0 | 0 | 0 | 0 | B | 0 | 0 | B |
| C | 0 | 0 | 0 | 0 | 0 | 0 | 0 | C | 0 | C |
| D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | D | C |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

considered, that carry out the following transformations:

$$R\, d_i = d_{i+1}\ (i \bmod n) \qquad R\, D_i = D_{i+1}\ (i \bmod n)$$

$$N\, d_i = D_{i+1}\ (i \bmod n) \qquad N\, D_i = d_{i+2}\ (i \bmod n).$$

We need to identify the function $g(x, y)$. For that purpose, we will successively consider the four "quadrants" in which it is organized according to the dialectic values. The diagonal immediately follows. Due to property I, then $d_i \Rightarrow d_i = d_i$ and $D_i \Rightarrow D_i = D_i$. The remaining cases are based on the property that the sum of any two maximum elements in the lattice equals 1.

- *Cuadrant D–D*. If the same reasoning used in **Dn** for maximum and minimum elements is used and we have that $D_0 \Rightarrow D_k = D_0 \Rightarrow R^k\, D_0$, then $R^{-k}\, D_0 \Rightarrow D_0$, Then, due to ED, it follows that $1 = D_0 + R^{-k}\, D_0 \Rightarrow D_0$ would be a thesis against $f_3 = 0$. As a consequence, $D_0 \Rightarrow D_k = 0$ for every $k \neq 0$. As a result, applying the rotation, $D_j \Rightarrow D_k = 0$ for every $j \neq k$.
- *Cuadrant d–d*. Let us consider $d_j \Rightarrow d_k$. By applying MTE, we obtain $N\, d_k \Rightarrow N\, d_j = D_{k+1} \Rightarrow D_{j+1} = 0$, then $d_j \Rightarrow d_k = 0$ for every $j \neq k$.
- *Cuadrant D–d*. Let us assume that $D_0 \Rightarrow d_i$ is a thesis. By applying MTE, we have that $N\, d_i \Rightarrow N\, D_0 = D_{i+1} \Rightarrow d_2$ which

is also a thesis. By applying the rotation $R^{2-i}$ to the first implication, we obtain $R^{2-i}\,D_0 \Rightarrow R^{2-i}\,d_i = D_{2-i} \Rightarrow d_2$, which is also a thesis. Due to DE we have that $D_{i+1} + D_{2-i} \Rightarrow d_2$ which is also a thesis. But the sum of two different maximum elements equals 1, then for $i + 1 - (2 - i) = 2\,i - 1 = \pm 1, \pm 3, \pm 5 \cdots$ they are theses. For $i = 0, 1$ we obtain $1 = D_2 + D_1 \Rightarrow d_2 = 0$ and because $f_3 = 0$, then the starting hypothesis is false. Ultimately, $D_0 \Rightarrow d_1 = 0$ and $D_0 \Rightarrow D_0 = 0$. Due to the rotation, $D_j \Rightarrow d_{j+1} = 0$ and $D_j \Rightarrow d_j = 0$. For $i = 2, -1$ we obtain $D_0 \Rightarrow d_2 = 0$ and $D_0 \Rightarrow d_{-1} = 0$ and due to the rotation, $D_j \Rightarrow d_{j+2} = 0$ and $D_j \Rightarrow d_{j-1} = 0$. For $i = 3, -2$ we obtain $D_0 \Rightarrow d_3 = 0$ and $D_0 \Rightarrow d_{-2} = 0$ and due to the rotation, $D_j \Rightarrow d_{j+3} = 0$ and $D_j \Rightarrow d_{j-2} = 0$ and so on. Then, all the elements verify that $D_j \Rightarrow d_k = 0$.

- *Cuadrant d–D*. The demonstration is similar to that of the previous quadrant. We will assume that $d_0 \Rightarrow D_i$ is a thesis. By Applying MTE, we have that $d_{i+2} \Rightarrow D_1$ which is also a thesis. By applying the rotation $R^{1-i}$ to the first implication, we obtain $d_{1-i} \Rightarrow D_1$ which is also a thesis. Due to DE we have that $d_{i+2} + d_{1-i} \Rightarrow D_1$, which is also a thesis. But the sum of two atoms with indexes that are separated by 2 or more is worth 1, which contradicts $f_3 = 0$, Then, the starting hypothesis is false. Then, for $i + 2 - (1 - i) = 2\,i + 1 = \pm 3, \pm 5, \cdots$ they are theses. For $i = 1, -2$ we obtain that $d_0 \Rightarrow D_1 = 0$ and $d_0 \Rightarrow D_{-2} = 0$. For $i = 2, -3$ we obtain that $d_0 \Rightarrow D_2 = 0$ and $d_0 \Rightarrow D_{-3} = 0$ and so on. Then, by rotation, $d_i \Rightarrow D_{i+1} = 0$, $d_i \Rightarrow D_{i+2} = 0$ and similar ones also occur. $d_i \Rightarrow D_{i-2} = 0$, $d_i \Rightarrow D_{i-3} = 0$ and similar are null. Nothing is known of $d_i \Rightarrow D_i$ or $d_i \Rightarrow D_{i-1}$.

- If we consider the thesis $d_i \Rightarrow d_i$, $d_i \Rightarrow d_i + d_{i+1} = D_i$ and $d_i \Rightarrow d_i + d_{i-1} = D_{i-1}$ are also theses. To meet PM, it must occur that $d_i \Rightarrow D_i$ worth either $d_i$ or $D_i$, and the same thing happens in the other case.

This completes the demonstration. □

If we omit the PM property–a mixture of the dialectic values–in quadrant d–D, the values can be replaced by others obtained by rotation. Thus, for instance, in Table 35 the value can be replaced by $a \Rightarrow A = B$ and the corresponding values by rotation. This function complies with all the formal and semantic properties except for PM.

It is worth noting that for the demonstration of this theorem, the properties RI, I, PM, DE and MTE have been used directly and MTE, MP and DE have been used to prove that $f_3 = 0$. Something similar occurs with the demonstration in **Dn**. In these two cases, then, RI, I, PM, MP, DE and MTE are enough to verify that the other formal and semantic properties are met, something which is also suggested by Figure 26. It is a reasonable conjecture that this result is general for all dialectic lattices, but we will not delve in this matter since it would only be of interest from a theoretical standpoint and does not contribute much to dialectic logic itself.

## Implications in **3Dn**

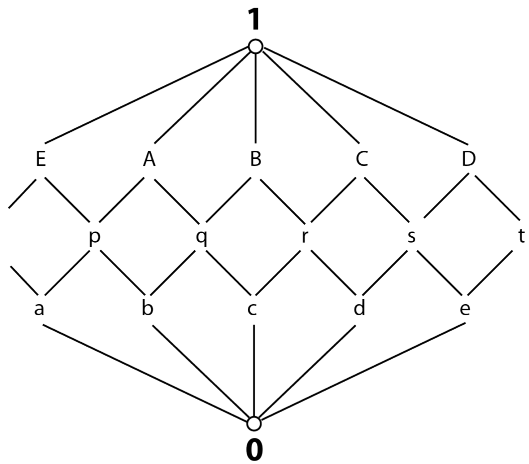


Figure 27: Lattice **3D5** as an example of **3Dn**.

It is worth considering the implication function in lattice **3D5**, see Figure 27, since it presents a major variation of the truth table. Table 36 presents the truth table that corresponds to the implication defined by the order relation in the lattice.

Table 36: Table of the minor implication in **3D4**.

| ⇒ | 0 | a | b | c | d | e | p | q | r | s | t | A | B | C | D | E | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | a | b | c | d | e | p | q | r | s | t | A | B | C | D | E | 1 |
| a | 0 | a | 0 | 0 | 0 | 0 | a | 0 | 0 | 0 | a | a | 0 | 0 | 0 | a | a |
| b | 0 | 0 | b | 0 | 0 | 0 | b | b | 0 | 0 | 0 | b | b | 0 | 0 | 0 | b |
| c | 0 | 0 | 0 | c | 0 | 0 | 0 | c | c | 0 | 0 | 0 | c | c | 0 | 0 | c |
| d | 0 | 0 | 0 | 0 | d | 0 | 0 | 0 | d | d | 0 | 0 | 0 | d | d | 0 | d |
| e | 0 | 0 | 0 | 0 | 0 | e | 0 | 0 | 0 | e | e | 0 | 0 | 0 | e | e | e |
| p | 0 | 0 | 0 | 0 | 0 | 0 | p | 0 | 0 | 0 | 0 | p | 0 | 0 | 0 | p | p |
| q | 0 | 0 | 0 | 0 | 0 | 0 | 0 | q | 0 | 0 | 0 | q | q | 0 | 0 | 0 | q |
| r | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | r | 0 | 0 | 0 | r | r | 0 | 0 | r |
| s | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | s | 0 | 0 | 0 | s | s | 0 | s |
| t | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | t | 0 | 0 | 0 | t | t | t |
| A | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | A | 0 | 0 | 0 | 0 | A |
| B | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | B | 0 | 0 | 0 | B |
| C | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | C | 0 | 0 | C |
| D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | D | 0 | D |
| E | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | E | E |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

This result illustrates the minor basic implication in all the **rDn** lattices. The major case can be defined in a similar manner.

The demonstration that the formal and semantic properties lead to a basic implication is somewhat more complicated than in the case of **2Dn**. Although in general terms the arguments for the quadrants formed by atoms and maximum elements apply, three new quadrants appear here: for central elements and central elements with atoms and maximum elements, in which a new argumentation must be developed. As in the previous cases, this argumentation is strongly based on formal properties MTE and DE and in the properties of the rotation. The demonstration will not be expanded here.

## Implication and the CI property

This section gives special consideration to the conjunction introduction property, IC, a property that does not meet the dialectic implication that we have studied. Regardless of this, it is possible to construct theories without compliance with CI being a restriction towards the rules of deduction.

Let us now consider a theory in which axioms have logical values belonging to a *cone* **S**, see Definition 14. It is clear that all the theorems in this theory will also belong to **S** due to the principle of mixture, PM. In fact, the principle of mixture keeps the implication function from constructing a theorem outside of the cone **S**, given that the dialectic values are not mixed together. This situation offers an important semantic clue on dialectic implication.

For these logical systems, the property CI of the implication is valid, since if two statements belonging to the theory have logical values $x, y \in$ **S**, then the statement $x \,.\, y \in$ **S**. In this way, we can assert that any theory whose axioms have logical values that belong to a cone, comply with all the formal properties of implication.

In conclusion, every argument that proves to be valid when using formal rules is also valid for the dialectic values in a cone. This allows us to make generalizations of the mathematical theories and those of the natural sciences–which are strongly supported by logical arguments–to dialectic values, without even the slightest alteration. This topic is analyzed in detail in the final chapter of this study.

# The dialectic of predicates

## Introduction

A propositional function is an application from the real universe–or a fragment of this universe–onto a lattice. According to traditional logic, propositional functions are *abstract functions*. In materialist dialectics, propositional functions represent *material properties*. For this reason, we will add the term "material" to most statements on propositional functions.

As in traditional logic, one-variable propositional functions may be referred to as *properties* and many-variable propositional functions as *relations*. We will consider the simple proposition

Lope **loves**.

In reality, this proposition is a material instance of property $F(u)$ of the material variable $u$ that covers the group of human beings or any other desired group. The property can be expressed as:[131]

$F(u) = u$ **loves**.

It is interesting to research the logical values that this proposition function acquires. In binary logic, given that there are only two logical values, we have two options. In the field of dialectics, it is an entirely different scenario. In this case, it is worth wondering what values the propositional function may take on. It is understandable that this function would fail to take a "true" logical value for any $x$, since this fact is almost impossible to interpret. We cannot understand how, upon taking the value "true", a proposition may be compatible with the material or actual meaning, for example, of the property "love". It would seem that–with the exception of some saint with a pathological incapacity to

[131] It is worth noting that this proposition is different–although similar to–the *relation* $P(u, v) = u$ **loves** $v$.

hate–, for common people, feelings are all mixed up and one is capable of "not loving" either at times (temporary logic) or by degree (modal logic).[132] Practically the same reasons can be argued for the logical value "false".

Let us think now of a different problem: what is the opposite property of $F(x)$? The answer is complex and multi-faceted. As noted earlier, the following propositions

$H(u) = u$ **hates**

$I(u) = u$ **is a religious fanatic**

$J(u) = u$ **is out of his mind**

$K(u) = u$ **is dead**

$L(u) = u$ **is a character in a work of fiction**

and many similar ones that are clearly linked to the emotional state of the individual, are all properties that oppose $F(u)$ in some way, according to their material meaning. This is why we are interested in characterizing the dialectic property of the material contradiction in a precise manner.

**Definition 33** *Two properties $F(u)$ and $G(u)$ are referred to as* material opposites *if, for some defined negation $N$, for every value $x$ of the material variable $u$, $F(x) = N\,G(x)$.*

The notion of material opposites is a basic notion in dialectics. The concept of property is usual in mathematics. In this case, it is a propositional function that only takes on the values "true" or "false". The importance of extending this notion to dialectics stems from the fact that the function may acquire dialectic values. In fact, most of the properties we use in everyday language only take on dialectic values, because nothing in our everyday lives is entirely "true" or "false", but rather something in between.

[132] It would also appear to be a poetic license–used in a famous sonnet by Quevedo–that a person can "love" beyond death.

Hegelian logic–and the more complex dialectics–are able to resolve this issue. Given that Hegelian dialectics have three dialectic values, any propositional function that acquires dialectic values has a large number of opposing properties. For every dialectic value, there are two opposed dialectic values. As a result, a property that only acquires dialectic values, such as $F(u)$, has a large number of opposite functions. As an example, a Hegelian function such as $F(u)$, defined over a set that has $n$ material instances–all the human beings, for instance–(potentially) has $2^n$ opposite propositional functions. The existence of more than two values that are opposite between themselves is the quantum leap that provokes the change in quality. It is, once again, the application of the laws of dialectics.

## Classic quantifiers

In binary logic, the notion of a quantifier is an essential element in the study of propositional functions. These ideas extend to dialectics in a straightforward manner since the operations "." and "+" have already been defined. In any given lattice, quantifiers extend the properties of the existential and the universal quantifier, which, just like in binary logic, is formally defined.

**Definition 34** *In every dialectic lattice, if $F(u)$ is a propositional function in the variable $u_i \in$ **C**, the* existential *and* universal *quantifiers can be defined as*

$$\exists u\, F(u) = F(u_1) + F(u_2) + \ldots \qquad \forall u\, F(u) = F(u_1) \,.\, F(u_2) \,.\, \cdots$$

*extensive to the values of the material variables that are being quantified and $u$ representing a set of material variables $u = (x, y, \ldots)$.*

The following theorem analyzes the behavior of the universal quantifier in a dialectic lattice.

**Theorem 59** *The necessary and sufficient condition for $\forall u\, F(u)$ to be 0 is that some instance of the function $F(u)$ be 0. The necessary and sufficient condition for the universal quantifier to be 1 is that all the instances of the function be 1. The necessary and sufficient condition for the universal quantifier to be a dialectic value $d$ is that all the values in the function belong to a* cone $d > 0$.

**Proof**. For a product to be 0, at least one factor must be 0. Reciprocally, if a factor is 0, the product is null. For a product to be 1, it is necessary that all factors be 1 and this condition is sufficient. If $\forall u\, F(u) = d$ is a dialectic value $d > 0$, then every value $F(u_i)$ verifies that $F(u_i) \,.\, (\forall u\, F(u)) = \forall u\, F(u)$ due to the application of the commutative, associative and idempotent properties. Then, $F(u_i) \,.\, d = d$, that is $F(u_i) \geq d$ and belongs to the cone of the elements such that $x \geq d$ as needed to be proven. Reciprocally, if for every $i$, $F(u_i) \geq d$, where $d > 0$ then $\forall u\, F(u) \geq d > 0$ and it is a thesis. □

The following theorem analyzes the dialectic existential quantifier.

**Theorem 60** *The necessary and sufficient condition for $\exists u\, F(u)$ to be 0 is that all instances of the function $F(u)$ be 0. The necessary and sufficient condition for the existential quantifier to be 1 is that some instance of the function be 1. The necessary and sufficient condition for the existential quantifier to be a dialectic value is that all the values in the function belong to an* inverted cone $d < 1$.

**Proof.**[133] For a sum to be 0, all the summands must be 0 and reciprocally. For a sum to be 1, it is enough for an instance of the function to be 1 and reciprocally. If $\exists u\, F(u) = d$ is a dialectic value $d < 1$, then $F(u_i) + \exists u\, F(u) = \exists u\, F(u)$ due to the application of the commutative, associative and idempotent properties. Then, $F(u_i) + d = d$, that is to say $F(u_i) \leq d < 1$ and belongs to the inverted cone of the elements such that $x \leq d$ as needed to be proven. Reciprocally, if for

[133] It is possible to demonstrate this by applying Theorem 65 directly.

every $i$, $F(u_i) \leq d$, where $d < 1$ then $\forall u\, F(u) \geq d < 1$ and it is a thesis. □

This definition generalizes the quantifiers in binary logic and preserves the fundamental semantics of existence and universality, as shown in the following theorem.

**Theorem 61** *For the quantifiers $\forall, \exists$ the following holds true, if $p$ is an instance of the variable $u$:*

$$\forall u\, F(u) \Rightarrow F(p) \qquad F(p) \Rightarrow \exists u\, F(u).$$

**Proof**. The following stems from the definition of quantifier, properties A, C and the monotony of the product, for every lattice value

$$\forall u\, F(u) = F(u_1)\,.\,\cdots\,.\,F(p)\,.\,\cdots \leq F(p)$$

given that $p$ is one of the instances of variable $u$. Then $\forall u\, F(u) \Rightarrow F(p)$ by Definition 32. In a dual manner, the following holds true

$$F(p) \leq F(u_1) + \cdots + F(p) + \cdots = \exists u\, F(u)$$

Then $F(p) \Rightarrow \exists u\, F(u)$. □

## Dialectic quantifiers in general

Extending the definition of classic quantifiers to dialectic logic lattices not only follows from the formal definition but also includes the expected functional properties. In this sense, the logic of predicates can be generalized, just as a generalization can be made based on implication. However, there is good reason to think that this method of operation leaves out many features from dialectics. In his *Science of Logic* [42], Hegel devotes a large volume to what he refers to as the "theory of being". This fact alone should warn us that the "theory of being" (or dialectic quantifiers, from a formal point of view) can be quite more complex than merely extending the notions behind binary logic.

We will use a formal methodological path to analyze the problem of quantifiers. As we have considered before, binary logic is an oversimplification, an all-too-radical homomorphism of the structural properties of the Universe. Because of this simplification, binary logic can only offer clues as to the problem.

According to this, we can elaborate the general definition of a *dialectic quantifier*.

**Definition 35** *The* quantifier ꓘ *of the propositional function* $F(u)$, *associated with the* non-trivial *dialectic operation represented by* $\diamond$, *idempotent, associative, commutative, rotationally invariant (I, A, C, RI), aside from the monotony properties BP and DP and of permanence of the binary rules PP, is referred to as the expression:*

ꓘ $u\,F(u) = F(u_1) \diamond F(u_2) \diamond \ldots$

*extensive to all the values of the material variables on which the quantification is being done. The variable* $u$ *can represent a set of material variables* $u = (x, y, \ldots)$.

In this definition, the *trivial* operation $\diamond$ ($x \diamond x = x$, and the remaining values 0) meet the properties I, A, C, RI, BP and DP but lack any application of interest.

As immediately follows, this definition generalizes the one made by binary logic and contains the existential and universal quantifiers defined in binary logic as specific cases. In fact, given that the operations "." and "+", meet the properties I, A, C, RI, BP and DP, the two quantifiers–respectively $\forall$ and $\exists$–are comprised within the definition. It is possible to demonstrate the inverse result.

**Theorem 62** *The only non-trivial, dialectic quantifiers in the binary lattice **B** = **D0** are those that derive from the conjunction and disjunction operations.*

**Proof**. The demonstration amounts to noting that only the functions I, A, C, RI out of the 16 functions that are possible in this lattice, are the ones indicated. □

Theorem 62 ensures the coherence of the definitions, but still leaves a great deal of open terrain.[134]

**Theorem 63** *If we consider a quantifier* $\reflectbox{$\lambda$}$ *and its associated composition operation* $\diamond$*, the following property holds true*

$$\reflectbox{$\lambda$}\, u\, (F(u) \diamond G(u)) = \reflectbox{$\lambda$}\, u\, F(u) \diamond \reflectbox{$\lambda$}\, x\, G(u)$$

**Proof**. It follows immediately based on the associative and commutative property of the composition $\diamond$. □

Definition 35 establishes a clear connection between penetration functions and quantifiers. The panorama for quantifiers is now completely defined. There are three major groups associated with the ideas of "being" and these sets are linked to the major groups of logical functions: the basic lattice operations and penetrations. It is natural, then, for two families of dialectic quantifiers to exist, *ample quantifiers* originated in ample penetrations, and *strict quantifiers*, originated in strict penetrations. We will study these cases in the following sections.

## Ample dialectic quantifiers

One type of quantifier is lost entirely in the simplification made by binary logic. We use the notation $\forall$ and $\exists$ with the meaning given in Definition 34 for classic quantifiers, and $\reflectbox{$\lambda$}, \reflectbox{$\lambda$}_n$ respectively, for those obtained by means of the ample penetration functions $*, *_n$.

[134] Therefore, for example, in **Dn** lattices, there are four possible non-trivial functions: conjunction, disjunction and the two penetration functions.

**Theorem 64** *For quantifiers* $\rightthreetimes$ *and* $\rightthreetimes_n$ *and every propositional function, the following relations are met:*

$$\forall u\, F(u) \leq \rightthreetimes u\, F(u) \leq \exists u\, F(u)$$

$$\forall u\, F(u) \leq \rightthreetimes_n u\, F(u) \leq \exists u\, F(u)$$

*extensive to all the values of the material variables on which the quantification is being done. The variable* $u$ *can represent a set of material variables* $u = (x, y, \ldots)$.

**Proof**. The properties follow immediately based on the monotony of the functions involved. In fact, if we consider the case of $*$, by definition

$$\rightthreetimes u\, F(u) = F(u_1) * F(u_2) * F(u_3) * \cdots$$

It is clear that the following is met, due to the associative property and due to BP and DP

$$F(u_1) * (F(u_2) * F(u_3) * \cdots) \leq F(u_1) + (F(u_2) * F(u_3) * \cdots)$$

It is also clear that

$$F(u_2) * (F(u_3) * \cdots) \leq F(u_2) + (F(u_3) * \cdots)$$

By repeatedly applying these inequalities, the following occurs

$$F(u_1) * F(u_2) * F(u_3) * \cdots \leq F(u_1) + F(u_2) + F(u_3) + \cdots$$

with which it is proven that $\rightthreetimes u\, F(u) \leq \exists u\, F(u)$. In a dual manner, it is proven for the product $\forall u\, F(u) \leq \rightthreetimes u\, F(u)$. Then, the theorem holds true for the quantifier $\rightthreetimes$. The same relations are met for $\rightthreetimes_n$ given that $*_n$ complies with the same inequalities as $*$, thus proving the theorem. □

The connection between negations and $*, *_n$ is extensive to quantifiers.

**Theorem 65** *For every quantifier* $\text{⅄}$ *and every negation* $N$*, the following holds true:* $N \text{⅄} u\, F(u) = \text{⅄}_n u\, NF(u)$ *and, in a dual manner, interchanging* $\text{⅄}$ *and* $\text{⅄}_n$*. Analogously, the following holds true:* $N \forall u\, F(u) = \exists u\, NF(u)$ *and, in a dual manner, interchanging* $\forall$ *and* $\exists$*.*

**Proof**. If we consider the expression, which is valid due to the associative property of $*$,

$$
\begin{aligned}
N \text{⅄} u\, F(u) &= N(F(u_1) * F(u_2) * F(U_3) * \ldots) \\
&= N(F(u_1) * (F(u_2) * F(U_3) * \ldots)) = \\
&= NF(u_1) *_n N((F(u_2) * F(U_3) * \ldots)) = \\
&\ldots \\
&= NF(u_1) *_n NF(u_2) *_n NF(U_3) *_n \ldots = \text{⅄}_n u\, NF(u)
\end{aligned}
$$

The theorem is proven by recurrence. The dual case is proven in an equal manner. In the case of classic quantifiers, the demonstration is the same, replacing the penetration functions with the sum and the product. □

The definitions made allow us to research the basic properties of ample quantifiers by extending the properties of binary logic.

**Theorem 66** *The necessary and sufficient condition for* $\text{⅄}_n u\, F(u)$ *to be a thesis is that all the values* $F(u_i)$ *are theses.*

**Proof**. For $\text{⅄}_n u\, F(u) = 0$, some $F(u_i) = 0$. Then, in order for the quantifier to be a thesis, all the values must be a thesis. Reciprocally, if all the values are theses, the result is not 0. □

This result shows that the quantifier $\text{⅄}_n$ can be referred to as “universal”, extending the binary notion of “true” to “thesis”, as is done in dialectics.

**Theorem 67** *The necessary and sufficient condition for the quantifier* $ꓘu\, F(u)$ *to be worth 1 is that at least one instance* $i$ *is* $F(u_i) = 1$. *The necessary and sufficient condition for* $ꓘu\, F(u)$ *to be a dialectic value is that for every* $i$, $F(u_i) \geq d$, *where* $d > 0$ *is a thesis value in the lattice. Equivalently, the values in the function belong to a non-trivial cone in the lattice. In the remaining cases, it is worth 0.*

**Proof**. Since $x * 1 = 1$ occurs, it is clear that if an instance of the function is worth 1, the quantifier is worth 1. Reciprocally, it is necessary for some instance to be worth 1 for the result to be 1. It is clear that if for every $i$, $F(u_i) \geq d$ holds true, then, $d \leq \forall u\, F(u) \leq ꓘu\, F(u)$, (Theorem 64) and the quantifier is a thesis. Reciprocally, if $ꓘu\, F(u) = d$ then $0 < d < 1$ holds true. Then, every $F(u_i) \neq 1$ and therefore it occurs that $\forall u\, F(u) = ꓘu\, F(u)$ (see Definition 26) and Theorem 59 applies. The quantifier is worth 0 in the remaining cases. □

The symmetrical quantifier meets a theorem that is also symmetrical.

**Theorem 68** *The necessary and sufficient condition for it to be worth 0 is that at least one instance* $i$ *be* $F(u_i) = 0$. *The necessary and sufficient condition for* $ꓘu\, F(u)$ *to be a dialectic value is that for every* $i$, $F(u_i) \leq d$, *where* $d < 1$ *is a dialectic value in the lattice. Equivalently, the values in the function belong to a non-trivial inverted cone in the lattice. In the remaining cases, it is worth 1.*

**Proof**. Let us consider the function $N^{-1}F(u)$ and the quantifier $ꓘu\, N^{-1}F(u) = N^{-1}ꓘ_n u\, F(u)$ due to Theorem 65. By applying the negation to the equality, we have that $ꓘ_n u\, F(u) = N\, ꓘu\, N^{-1}F(u)$. Theorem 67 allows us to demonstrate the conditions. For $x = ꓘ_n u\, F(u)$ to be worth 0, it must occur that $ꓘu\, N^{-1}F(u)$ be worth 1, then at least one instance $i$ of $N^{-1}F(u_i) = 0$ to be worth 0, it must occur that $F(u_i) = 1$. For the value of the quantifier to be dialectic, for every $i$, $N^{-1}F(u_i) \geq d$, where $d > 0$ is a thesis value, that is,

$F(u_i) \leq N\, d = d' < 1.$[135] The quantifier is worth 1 in the remaining cases.□

Ample quantifiers behave differently with regards to the properties in Theorem 61, as shown in the following theorem.

**Theorem 69** *For the ample quantifier* ʎ *, if* $p$ *is an instance of the variable* $u$*, it occurs that* ʎ $u\, F(u) \Rightarrow F(p)$*, provided the quantifier is not worth 1.*

**Proof**. Given $d =$ ʎ $u\, F(u)$. If $d = 0$, the theorem is valid in a trivial manner. If $1 > d > 0$ then $F(p) \geq d$ is met due to Theorem 67, then the implication is met. If $d = 1$, there are cases where the expression is not valid, it suffices with $F(p)$ being either 0 or dialectic. □

**Theorem 70** *For the ample quantifier* ʎ$_n$ *if* $p$ *is an instance of variable* $u$*,* $F(p) \Rightarrow$ ʎ$_n u\, F(u)$*, holds true, as long as the quantifier is not worth 0.*

**Proof**. Given $d =$ ʎ $u\, F(u)$. If $d = 1$ the theorem is valid in a trivial manner. If $1 > d > 0$ then $F(p) \leq d$ holds true due to Theorem 68, then the implication is met. If $d = 0$, there are cases in which the expression is not valid, it suffices with $F(p)$ being either 1 or dialectic. □

Table 37 presents the main properties of the dialectic quantifiers analyzed in the last sections.

## Strict dialectic quantifiers

As we have already analyzed, there is a second type of penetration function that we have referred to as strict, that only occurs in odd-rank

[135] The theorem can also be proven in a direct manner based on the definition of the quantifier. This demonstration illustrates how to employ the existing duality between ʎ and ʎ$_n$.

Table 37: Summary of dialectic quantifiers.

| | $\exists$ | $\forall$ | ꓘ | ꓘ$_n$ |
|---|---|---|---|---|
| false | $\forall i\, F(u_i) = 0$ | $\exists i\, F(u_i) = 0$ | other cases | $\exists i\, F(u_i) = 0$ |
| thesis | $\forall i\, F(u_i) \leq d$ $d < 1$ | $\forall i\, F(u_i) \geq d$ $d > 0$ | $\forall i\, F(u_i) \geq d$ $d > 0$ | $\forall i\, F(u_i) \leq d$ $d < 1$ |
| true | $\exists i\, F(u_i) = 1$ | $\forall i\, F(u_i) = 1$ | $\exists i\, F(u_i) = 1$ | other cases |

lattices. The corresponding quantifiers are associated with these functions. The only case of interest is for penetrations $\bar{*}^d$, where the following theorem holds true.

**Definition 36** *The* strict quantifier $\bar{\text{ꓘ}}^{\,d}$ *of the propositional function $F(u)$, associated with the strict dialectic penetration $\bar{*}^d$, idempotent, associative, commutative, rotationally invariant (I, A, C, RI), aside from the properties BP and DP, is referred to as the expression:*

$$\bar{\text{ꓘ}}^{\,d} u\, F(u) = F(u_1)\, \bar{*}^d\, F(u_2)\, \bar{*}^d \ldots$$

*extensive to all the values of the material variables on which the quantification is being done. The variable $u$ can represent a set of material variables $u = (x, y, \ldots)$.*

These quantifiers have different properties than ample quantifiers. The most important property occurs when the values of the material variables belong to a *cone*.

**Theorem 71** *The strict quantifiers of a propositional functionl $F(u)$ take on thesis value if and only if all the instances $u_i$, of the function are within a cone $u_i \geq d > 0$ inside the lattice. They are worth 1 only if all the instances meet $F(u_i) = 1$. In all the remaining cases, they are worth 0.*

**Proof**. Proof. If they are in a cone such as the one indicated, the quantifier is a thesis. Reciprocally, for the quantifier not to be null, all the instances must be in a cone such as the one indicated. For the result of the quantifier to be worth 1, the only possibility, in the case of a penetration $\bar{*}^d$, is that all the instances be worth 1. □

## The semantics of dialectic quantifiers

The contribution of dialectic quantifiers to the general logic is varied, yet has no actual relevance for the study at hand. To begin with, existential and universal quantifiers *extend the notions of binary logic.* This extension is quite significant when analyzing the applications of dialectics in the sciences. As it becomes accepted that mathematical and scientific theories may take on dialectic values, the properties of implication and the logic of propositional functions must also be extended with the same formal properties. We will analyze this in the final chapter.

This brings about the question: what is the contribution of the new dialectic quantifiers as derived from penetration functions? The answer to this does not come from science, but from the spontaneous logic of natural languages. We will go over the problem of the "illogicality" of the definition of love, see page 19.

According to our analysis, the sonnets by Petrarca, Lope and so many others, are not a dialectic quantifier on human passions—especially those defined with regards to strict penetrations. If we go back to Lope de Vega's description from page 129:

> (to faint, to dare, to be enraged), (coarse, tender), (liberal, elusive, encouraged), (mortal, dead, alive), (loyal, traitor), (coward and brave).

In order to formalize this description, we need to introduce several propositional functions to be applied on the universe of human beings $u$, such as:

$P_1(u) = u$ **faints**

$P_2(u) = u$ **dares**

$P_3(u) = u$ **is enraged**

This example of three human states illustrates the idea. Each of these propositional functions describes a *passion*[136] and they apply to all human beings. At the same time, they cannot but take on a dialectic value: no one can be entirely enraged–we may be enraged at times and calm, indifferent or sleeping at other moments, just to mention a few possibilities. Let us, then, accept that these propositional functions can only take on dialectic values.

If move on to the subject of the commas, as we have previously analyzed, the comma represents a logical function with the properties I, A, C and with an intermediate value between conjunction and disjunction. In other words, a penetration function. With these considerations, the first parenthesis in Lope's definition can be formalized as

$$P_1(u) \,\bar{*}^d\, P_2(u) \,\bar{*}^d\, P_3(u).$$

It is legitimate to wonder why we should use a strict, rather than an ample, penetration. There are various possible reasons for this. The first one may be that ample penetrations have a dual version obtained by negation. It seems reasonably clear that the passions referred to in this case lack a well-defined negation and this is one of the reasons. A second reason is found in the use of pairs or triads of passions and this strongly points to a strict penetration, which better suits the condition of an intermediate state between two opposite situations.[137]

The same technique can be applied to the other pairs or triads of human passions. Thus, for example, we could define:

$P_4(u) = u$ **is coarse**

$P_5(u) = u$ **is tender**

and the formalization of the description as

$$P_4(u) \,\bar{*}^d\, P_5(u).$$

[136] Oxford [72] defines the word passion as follows: strong and barely controllable emotion; a state or outburst of strong emotion; intense sexual love; an intense desire or enthusiasm for something; a thing arousing enthusiasm.

[137] Regardless of these considerations, any of the defined quantifiers–and this includes the classic quantifiers–meet this property. Strict quantifiers lead to more symmetrical values.

Then, what is left for us to define is the function that replaces the commas that join the pairs and triads of passionate states together. It is once again clear that this function is I, A, C and that it has an intermediate value between conjunction and disjunction: it is yet another penetration function.

We must specify the use of the word "passion" before we go any further. It is clear that, aside from the human passions that refer to love, there are other human passions that have nothing to do with it. Page 193 offers other cases which are possibly contrary to love. Dialectic lattices give us tools for analyzing this situation. If we consider, for example, the cases

$I(u) = u$ **is a religious fanatic**

$J(u) = u$ **is out of his mind**

There is no doubt that there are many ways of being a *religious fanatic* or *being out of one's mind*. The different religions, current or past, display various examples of this.[138] Psychiatry shows very different possible states of behavioral anomalies ranging from autistic to serial killer, as examples of the second case.

As we have already proposed, these cases of human passions are, to a great extent, contrary to the passions aroused by love. From a dialectic standpoint, we must consider that they acquire opposite dialectic values. In a schematic manner, we might establish the following association within lattice **3Dn**:

- loving passions acquire values in the interval $(a, p, A)$,
- religious passions acquire values in the interval $(b, q, B)$,
- behavioral anomalies acquire values in the interval $(c, r, C)$.

and we could go on with other opposing passions. It seems clear that each group of human passions opposes all the other groups of passions, but we have no difficulty in assigning logical values to them.

[138] Without being a comprehensive list, there are Christian ascetics, Cathars or Templars in Christianity, Dervishes or Muslim martyrs in Islam, the fighters of the *Flower War* between the Aztecs and many others.

Now, the definition of love acquires a very simple and direct formal expression. According to the expression defined in page 192, we have that:

$$F(x) = P_1(x) \mathbin{\bar{*}^d} P_2(x) \mathbin{\bar{*}^d} P_3(x) \mathbin{\bar{*}^d} P_4(x) \mathbin{\bar{*}^d} P_5(x) \mathbin{\bar{*}^d} \cdots$$

which can also be stated, if we use the notation $P(i, x) = P_i(x)$, as

$$F(x) = \overline{\reflectbox{K}}^{d} i\, P(i, x).$$

The semantics and the use of the strict dialectic quantifier are thus explained and by extension, of all the defined dialectic quantifiers.

# Paradoxes

## Introduction to logical paradoxes

Paradoxes are usually one of the issues that pose the most problems to logicians. Every paradox holds a new truth within itself. Far from being an obstacle to a theory, it constitutes a source of new ideas–such is the creative power of contradiction. This way of regarding contradiction is, in essence, dialectic. Many authors have professed their admiration for paradoxes. For instance, W.K. Chesterton could not think if not in paradoxical terms. Wilde would rightly point out:

> Well, the way of paradoxes is the way of truth. To test reality we must see it on the tight rope. When the verities become acrobats, we can judge them.[96]

In binary logic, paradoxes can be sorted into two major types: paradoxes originating from propositional equations, and paradoxes stemming from functional equations. Their common background features an introduction that is within acceptable logical limits but leads to a contradiction. Classical binary logic can withstand anything but a contradiction, which leads to a problem. In dialectic logic, contradictions are not an issue.

Logical paradoxes are ruthless resolved by logicians: they state that the operational procedures leading to the formulation of contradictory equations are simply *not right*. There is usually much insistence on the notion of a meta level and the impossibility that logic may have a say on logic. The surgical solution of removing all nuisances prevents us from mining the richness contained within paradoxes.

In this chapter, we will consider some well-known paradoxes and only comment on others, without further study.

## The hanging or beheading paradox

A classic example of a paradox, whose origins are lost among medieval stories, is the problem of the man who has been sentenced to death, see [12, II, li]. It states that a man who has been punished with death is given a choice as to the way he wishes to die, with the caveat that, if he lies, he will be hanged and if he tells the truth, he will be beheaded. As it is not difficult to imagine, a clever convict will state that he will hang. Let us assume that we make the following propositional statements:

| | |
|---|---|
| $x$ | is the statement made by the convict |
| $a$ | the convict dies from beheading if he tells the truth |
| $b$ | the convict dies from hanging if he does not tell the truth |
| $V$ | is the set of truthful statements |

The convict has two choices:

| | |
|---|---|
| $(x \in V) \Rightarrow a$ | if he tells the truth, he dies from beheading |
| $N(x \in V) \Rightarrow b$ | if he does not tell the truth, he dies from hanging |

With these equations, the statement $x = b = N\,a$ leads to the expression:

$$(N\,a \in V) \Rightarrow a \quad \text{that is} \quad N\,a \Rightarrow a$$

Thus arriving at a contradiction which is unacceptable to binary logic, but is perfectly acceptable in dialectic logic. The statement "I will die from hanging" has *thesis* value but is not *true*. As currently interpreted, there is nothing mandatory or compulsive–while he may die in one way or the other, the truth is that he will die. Instead, to classical logic, the paradox stems from the fact that it cannot resolve how a person on death row may be saved because a set of equations has no solution and therefore, the method of execution cannot be decided.

The interesting thing is that the paradox can continue. Let us assume that the person on death row we have been referring to is saved because the logical paradox cannot be resolved. Society–having learnt from this case–subsequently introduces a new law to prevent this same situation from ever happening again.

| | |
|---|---|
| $x$ | is the statement made by the convict |
| $a$ | the convict dies from beheading if he tells the truth |
| $b$ | the convict dies from hanging if he does not tell the truth |
| $c$ | the convict dies from poisoning if he states a paradox |
| $V$ | is the set of truthful statements |
| $P$ | is the set of paradoxical statements |

With the new legal framework, the problem has now three social laws:

| | |
|---|---|
| $(x \in V) \Rightarrow a$ | if he tells the truth, he dies from beheading |
| $N(x \in V) \Rightarrow b$ | if he does not tell the truth, he dies from hanging |
| $(x \in P) \Rightarrow c$ | if he states a paradox, he dies from poisoning |

The convict now states that $x = c$ which leads to him not being able to die from poisoning since he has told the truth and thus fails to formulate a paradox, therefore $a$ is applicable. But then, he did not tell the truth, so $b$ applies. At the same time, it is clear that he has stated a paradox and so $c$ applies, but then we go back to the beginning, where he has told the truth. The contradiction persists. Even if new laws are passed, such as "if he states a second-order paradox, he is gunned down", the contradiction remains. In summary, dialectic poses no such contradiction and the obvious occurs: if he has been sentenced to die, he will die, regardless of the method of execution.

## Protagoras' paradox

A similar case occurs in Protagoras' paradox, see [85, X]. In this classic problem, Protagoras has trained a pupil in the art of litigation, under the condition that he gets paid whenever he wins a case. The paradox arises when the pupil refuses to pay for his education, which leads to Protagoras suing him. We then arrive at a case without a solution. Regardless of the result of the trial, one cannot logically conclude whether the pupil must or must not pay. Let us examine the problem through the following propositions:

| | |
|---|---|
| $a$ | Protagoras gets his pay |
| $b$ | The student wins a trial |
| $c$ | Protagoras wins the trial against his pupil |

The problem is very rich in statements useful to articulate all legal quibbles. To Protagoras, the possibilities are the following:

| | | |
|---|---|---|
| 1) | $b \Rightarrow a$ | contract: if he wins a trial, Protagoras collects money |
| 2) | $Nb \Rightarrow Na$ | contract: he does not win a trial, he does not collect money |
| 3) | $c \Rightarrow a$ | litigation: Protagoras wins and receives his pay |
| 4) | $c \Rightarrow Nb$ | litigation: Protagoras wins, indirect consequence |
| 5) | $Nc \Rightarrow Na$ | litigation: Protagoras loses, does not collect money |
| 6) | $Nc \Rightarrow b$ | litigation: Protagoras loses, indirect consequence |

It is easy to convince ourselves that these six equations have no solution within the realm of binary logic. On the other hand, in dialectic terms, it is a different story. To begin with, 1) and 2) are equivalent due to MTE. From 4), due to MTE, we have that $NNb \Rightarrow Nc$, that is, 7) $b \Rightarrow Nc$, which, combined with 5) due to T, gives us that 8) $b \Rightarrow Na$. From 1) and 8) due to PCE, we have that $Nb$ is $a$ thesis and also $b$ is a thesis. From 7) and 6) we have that $b$ and $Nc$ are equivalent, then $Nc$ is a thesis and also $c$ is a thesis. According to this, the problem has a solution and the three propositions take on dialectic values. Thus, "common sense" is resolved and we have a solution to the problem: the three propositions are theses and it follows that Protagoras receive his payment, whether he wins or loses in court.

## Epimenides' paradox

In its simplest form, Epimenides' paradox–or the liar's paradox–displays the most basic limitation of binary logic. In its classic form, see [83] for this and other paradoxes, the Cretan Epimenides is said to have stated

*all Cretans are liars.*

The paradox stems from the confrontation between this statement and the implicit statement

> *the author of the statement is a Cretan.*

There have been many other forms of this paradox, with varying degrees of complexity, but the allegedly original form contains all of the richness of the problem.

Let us begin with the methodological statement that most logicians will not accept: nothing can stop a person from imitating Epimenides by stating a phrase leading to the same paradox. This person's brain will not explode upon attempting this allegedly forbidden operation–nothing happens in our material world. This is where the true problem lies, that logicians tend to leave out. If the brain and the universe were *purely binary*, these statements could not be made, in much the same way as no one can walk on walls or circumvent the laws of thermodynamics. In different words, what is truly surprising in Epimenides' paradox is that there is no repugnance, no natural violence, no physical impossibility in stating it. Any reasonable person–even a professional logician–can understand the statement:

> *I lie*

despite it encompassing Epimenides' entire problem. It is simply absurd to assume that this everyday statement could be impossible–people often lie and they sometimes confess to it. Only a radical, idealistic position could imagine this statement being banned from the lives of people for being unthinkable. To dialectics, Epimenides' paradox contains an artificial trick that does not actually occur in real life.

It is not infrequent to say that the paradox arises out of confusion in the statements' hierarchies. From the moment a statement judges the validity of another, we can say that a level has been overcome and that we have gone from logic to meta-logic. This easy way of interpreting a paradox was promoted by Russell to help him escape his own paradox on classes and was popularized by Tarski to escape from other paradoxes.

This way of escaping from paradoxes is erroneous. As we can see in what follows, Epimenides' problem–because we cannot continue to refer to it as a paradox once we know they do not exist–is ultimately not in the mix of hierarchies but in attempting to find a binary solution to a non-binary logical problem. In fact [83] had already put forward this idea, although in a very basic form.

With the purpose of defining the analysis of Epimenides' problem, we must accept the following version, which is somewhat more precise:

1) $a$ the following statement is false,
2) $b$ the previous statement is true.

The paradox stems from assuming that statement $a$ is true, given that, then, $b$ would be false and from this, then $a$ is not true. Something similar occurs if we assume that statement $a$ is false. Since statement a cannot be either true or false, the alleged Epimenides' paradox occurs.

In a study of dialectics it is natural to state that $a$ has thesis value different from true and false. The problem can be formulated as: 1) $a \Rightarrow Nb$, 2) $b \Rightarrow a$, then, due to T, we have that $b \Rightarrow Nb$, then by PC $Nb$ is a thesis and then $b$ is also one. But let us analyze in more detail the steps leading to this.

Let us assume that we resort to the brain's spontaneous logic, without becoming trapped by artificial difficulties. It seems clear that the statements in Epimenides' problem can also be formulated as:

$a$ says that statement $b$ is false,
$b$ says that statement $a$ is not false.

So far, we have replaced true by the negation of false, which does not seem to set off any alarms. Let us consider the propositional function:

$$f(x) = \text{statement } x \text{ is false.}$$

With this function, Epimenides' problem becomes:

$$a = f(b) \qquad b = Nf(a).$$

The first statement says: $a$ establishes that $b$ is false. The second statement says: $b$ establishes that $a$ is not false. Epimenides' problem consists in studying whether these equations have a solution.

The classic solution is denying that the problem bears any significance. On the one hand, the function $f(x)$ must be a propositional function, but, on the other, it must be a logical function. This is the argument of confusion of levels that is usually invoked to escape from the paradox. But let us take on a wider criterion and move on. We will accept that $f(x)$ may be a logical function and that it is valid to have a say on the validity of a statement. In this case, the contradiction continues as follows. It is quite clear that $f(x)$ can only be one of the two only logical functions that exist: $f(x) = x$ or $f(x) = Nx$. It is reasonable to assume that we are referring to the second. If we accept that the function matches the first, which already evidences an inclination for the interpretation of the statement "$x$ is false", we arrive at the final equation $a = Na$.

With this interpretation, Epimenides' problem consists in solving the system of logical equations:

$$a = Nb \qquad b = NNa.$$

It is worth noting that we have not assumed that the negation is an involutory operation. If we were to replace $b$ in the first equation, we would arrive at $a = NNNa$. This equation can be solved in a myriad logical ways. Thus, for example, in Hegelian logic, $a$ can take on any of the three dialectic values, regardless of the negation considered. Even in second-order negation logics–which also occurs in some Hegelian negations–the resulting equation $a = Na$ has a solution. Therefore, for instance, in the modal logic defined in **C3**, there is a solution. In Hegelian logic, in **D3**, with the negation $N = (0\,1)$ there are three solutions. As surprising as it may seem, there are also solutions in Boolean logics of order greater than 1, for example, for the negation $N = (0\,1)$: it is clear that in *yin-yang* dialectics, both *yin* and *yang* are solutions for this negation. To sum up, the only problem that remains is deciding whether a system of logical equations has a solution within a certain logical environment, and nothing else. Even further, the solutions we have found allow us to translate the result obtained to a direct language: *Cretans only state strict theses, never truths or falsehoods* and this is the wonderful result that the alleged Epimenides' paradox has acquired. It is worth noting that if we wish to extract the meaning from the phrase "I lie" by means of a spontaneous attitude, we will arrive at the simple conclusion that the person making such statement is only worthy of partial credit. For example, his or her statements bear the stigma of doubt which is characteristic of modal logic, or the stigma of temporal validity, which is characteristic of Hegelian logic.

It is interesting to note that there is a very symmetrical way of stating Epimenides' problem, by means of three statements:

$a$ statement $b$ is false,
$b$ statement $c$ is false,
$c$ statement $a$ is false.

From here on, through an analysis similar to the previous, we have that $a$ must coincide with its multiple negation. We can continue through this procedure. As it is simple to understand, whenever we make an even number of statements, we will have a binary solution and the paradox will not even take place. Conversely, it suffices with the number being odd for the logical world to come crumbling down. This sensitivity to the parity of numbers bears no connection to the hierarchies of interpretation and meta-statements, but to the existence, or lack thereof, of solutions in a system of logical equations. It is difficult to believe that the trivial example of the existence of solutions in a system may be so central as to shake the very foundations of logic. Something similar happens in mathematics every time there is a problem that cannot be resolved, but the secular experience of mathematicians compels them to bravely venture into other numerical areas to find a solution. These adventures are known to have left deep scars in the field of mathematics: "irrational" and "imaginary" numbers evidence two clear wounds in the mathematical pride of those who ever wished to solve two simple quadratic equations. The exact same thing has happened in logic thanks to Epimenides' problem.

## Russell's paradox

In this section, we will study problems with a markedly functional nature. Among these, the so-called Russell's paradox distinctly stands out. Due to its importance from a theoretical point of view, this paradox marks a major focal point in the dialectical understanding of mathematics.

Let us begin our study at the point where the problem usually begins, in the so-called *barber paradox*. To this end, we will define the following propositional function:

$$F(x, y) = x \text{ shaves } y$$

This function is defined over the set of men (of a certain town, just to get a clear picture). Let $b$ be the town barber. When stating the problem, the barber shaves all those who do not shave themselves. This condition may be expressed as a truth table:

| | $F(x, x)$ | $F(b, x)$ |
|---|---|---|
| $x$ does not shave himself | 0 | 1 |
| $x$ shaves himself | 1 | 0 |

This table establishes the barber's double condition. The problem, thus stated, results in the propositional equation: $F(b, x) = NF(x, x)$. The paradox stems from applying this equation to the barber himself, since we would have: $F(b, b) = NF(b, b)$.

In binary logic, this equation does not have a solution. Not all the functional equations that we may come up with in the spur of the moment need to have a solution. This result in mathematics has been long-known. It is easy, then, to understand that the so-called paradox is nothing more than a problem without a solution, regardless of how clever and feasible the problem may be. The second aspect to consider is that in many a logic–for example modal logic–there is a solution for the problem and it establishes that "the barber shaves the barber" has thesis value. This solution is not a simple game of variables. Many definitions have been put aside when formulating the barber problem. We have all too lightly considered the issue of the number of barbers in the region and the absolutely truthful nature of the fact that there may be people who never shave themselves.

Let us now consider Russell's paradox, which is very similar to the barber problem. A class is defined by a property $p(x)$. For each individual $x$, we know whether property $p(x)$ is true or false (in the formulation made by binary logic). We will accept, just as Russell spontaneously accepted, that $x$ may also be a property. We can then study if the value of $p(p)$ is true or false. Then, we have the function $F(p) = Np(p)$ that expresses the property that $p$ does not contain the property $p$. We have thus constructed propositional function $F$, that comprises classes that do not contain themselves, according to the classical formulation. Let us now see the alleged paradox. Just as in the case of the barbers, we have a functional equation which may–eventually–not have a solution. Russell's problem occurs when $F$ is the property that we choose to study. We then arrive at a $F(F) = NF(F)$.

As we already know, this equation does not have a solution in binary logic–although it does in other types of dialectic logics–for instance, the "thesis" value. It is worth wondering if this answer leads to anything interesting or if it is simply a contrived outlet. Deep down, the problem lies in the fact that an element $x$ belongs to a class $p$ (value true), does not belong (value false) or belongs in a dialectic manner (thesis value). This is the reason why Russell's result is not contrived–instead of being an obstacle, it is a clear demonstration that the notion of class must be extended in a dialectic manner.

## Conclusions

The existence of propositional or functional equations without a solution within a certain logic does not constitute a paradox but a known mathematical problem. Mathematics have encountered this situation many times before. From the previous examples, we should not think that every problem has a solution within a certain dialectic logic. In propositional logic, every problem can be expressed as a system of equations of the type:

$$E_1 = v_1$$

$$\dots$$

$$E_p = v_q$$

where $E_i$ are logical expressions with a certain number of unknown propositions and it holds true that $v_1 = 0, 1$. We do not impose any type of restriction to the problem. In classical logic, to avoid the problem of reciprocal references, the possibility of writing the equality of two expressions or the mixing of variables is not accepted. But none of this keeps systems of logical equations that cannot be resolved from existing.

The first observation that can be made consists in considering that all the expressions are of the type $E_i = 0$ because an equation of the type $E_1 = 1$ is equivalent to $N\,E_1 = 0$. The second observation is that a system of expressions of which we ask that they be false is equivalent to:

$$E_1 + \cdots + E_p = 0.$$

With these observations, we can then prove that paradoxes can be formulated in every dialectic lattice and for every negation.

**Theorem 72** *In every lattice, for every $x, y, z$ and every negation $N$, the expression $p(x, y, z) = x + y + z \,.\, Nx + Nz \,.\, Ny$ is a thesis.*

**Proof**. For the expression to be 0, all the summands must be 0 and from this that the equations: $x = 0, y = 0, z.Nx = 0$ and $Nz.Ny = 0$ need to hold true. By replacing $x, y$ in the two remaining equations, we have that $z = 0$ and $Nz = 0$ which lack a solution in every lattice, for every negation. It is then proven that there is no triad of values for which the expression is worth 0, then, it is a thesis. □

As immediately follows, new variables may be added to the expression and it would continue to be a thesis–it is enough to add as many terms of the type $w \,.\, Nx + Nw \,.\, Ny$ as desired. Conversely, we can eliminate variables from the expression by assigning 0 values to $y$, with which we have that $x + z \,.\, Nx + Nz$ and also assigning 0 values to $z$, yielding $x + Nx$ which are theses in every lattice and every negation.

As another corollary to this theorem, in every lattice, for every negation and all logical functions, there are equations or systems of equations that do not have a solution. The equation $p(x, y, z) = 0$ is an example of this. They are what classical logicians refer to as *paradoxes.*

The existence of paradoxes suggests that, in dialectics, there is a whole new level of complexity under which these problems may be solved. This will be analyzed in the future.

# Dialectics in science

## Overview

In the first chapters of this book, we have shown that natural human thought uses logical structures that far exceed binary logic. This is the reason behind the subsequent development of many multi-valued, modal logics and other types of logics. In this study, we have proposed creating a formal structure–lattices, negations and logical functions–allowing us to formalize all of this rich logical content. This final chapter shows these structures being used in the formal, natural and social sciences.

Rank-1 lattices, **Dn**, allow us to understand simple contradictory statements. Figures 2, 3 and 5 contain examples of these lattices. Dialectic values can be interpreted as intermediate values between "true" and "false". This is needed to interpret Wilde's statements on art, certain love sonnets and most paradoxes. These lattices also enable us to naturally interpret the notion of *becoming of opposites*. Nothing keeps something from becoming its opposite as long as it holds dialectic value. Elements both in Ionia and China presented this property without breaching any of the rules of thought. Conversely, it is absurd for a mathematical theorem to be considered false, unless an error in its demonstration is proven, which would simply show that it *has been false all along*.

Rank-2 lattices, **2Dn**, help us understand that there are two types of opposites: *synchronic* and *diachronic* opposites. Figure 28 shows a case of these types of lattices. To begin with, in these lattices, the negation $\tilde{N}_0 = (0\,1)(a\,D)(b\,A)\cdots$ is present, which evidences the existence of synchronic opposites. One cannot exist without the other. They are bound by indissoluble pairs: *the unity and struggle of opposites*. But there is another negation $N_0 = (01)(a\,C\,d\cdots)(A\,b\,D\cdots)$ within the same lattice, relating diachronic opposites bound by *becoming*.

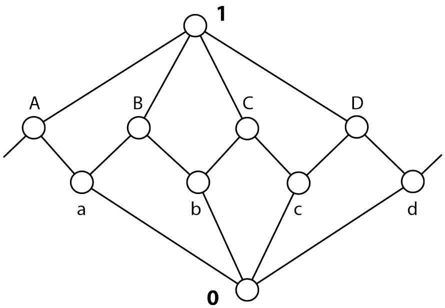

Figure 28: Generic rank-2 lattice.

This shows that, by using different negations, a single lattice–with the same "more truthful than" relations–can construct either static or dynamic interpretations of reality. Dialectic logic–just as binary logic–offers a method of interpretation. In much the same way as knowing about the rules of reasoning does not automatically generate mathematical knowledge, knowing about the formal rules of dialectic does not yield results on reality, by itself.

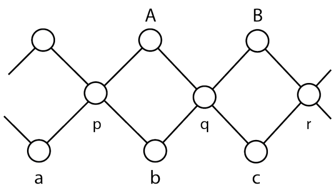

Figure 29: Generic rank-3 lattice.

Having explained this issue, the semantics in rank-3 lattices, **3Dn**, and by extension, those of a higher rank, immediately follow, as Figure 29 shows. As with **2Dn** the two types of opposites–synchronic and diachronic—exist in this lattice, with the addition of a supplementary element–element $p$ appears in the figure between $a$ and $A$, but there are even more intermediate elements in higher ranks, which are opposite among themselves—allowing us to interpret other concepts from historic materialism: the classes that stand between those that are opposite.

These more complex lattices show that, just like in the case of diachronic opposites, there can be more than two synchronic opposites.

A clear example occurs with *flavors*. The West identifies four synchronic opposites: sour, bitter, sweet and salty. The East–especially India–adds a further two: spicy and astringent. The seventh flavor, *umami*, is characteristic of fish, seafood and mushrooms and originated in Japan.

## Introduction to dialectics in the formal sciences

The formal sciences are characterized by an axiomatic structure that acts as the cornerstone of a purely deductive theory. There are as many formal sciences as there are possible sets of axioms; the only condition that is asked of axioms is that they do not contradict each other.

The non-contradiction of a set of axioms is far from a trivial matter. Only in the case of very few axioms can non-contradiction be demonstrated. The only reliable method for this proof consists in constructing a preferably *finite*, specific example–through an infinite example we would enter especially difficult ground–something which is not always possible or has been particularly accomplished.

At present, we are familiar with several groups of formal sciences:

- Binary logic and dialectic logic;
- Mathematics, which can be sorted into two major branches: discrete mathematics and continuum theory;
- The different geometries;
- Algorithm theory or symbol manipulation theory.

In discrete mathematics, it is fairly simple to find finite examples that comply with the axioms. Thus, for instance, the axioms in group theory are not contradictory because there are a large number of finite examples that fulfill the axioms. Continuum theory appears to be non-contradictory, but it is not free from some major theoretical difficulties, a couple of which are presented below. Geometries, based on the work of David Hilbert [46], are as free from contradiction as the continuum theory. Symbol manipulation theory—for delving into infinite problems—is as free from contradiction as natural number theory. It only *appears* to be non-contradictory.

Finally, because they are part of discrete mathematics, binary and dialectic logic quite simply present finite examples that fulfill the axioms of the former.

Independently from the lattice and negation being considered, one can always construct statements that are consistently thesis. This is a surprising result within the framework of dialectic logic, see Theorem 72.

## Contradiction in mathematics

In the formal sciences, contradiction plays a critical role. It is unacceptable and is only used as a method of demonstration. For this reason, we will begin by the *principle of contradiction*, also referred to as the *principle of explosion*.

Its classical Latin formulation, *ex contradictione quodlibet* (from a contradiction anything follows), has been well-known since the days of Scholastic logic.[139] In his critique of logicism, Henri Poincaré makes the following comment:

> *M. B. Russell arrive à cette conclusion qu'une proposition fausse quelconque implique toutes les autres propositions vrais ou fausses.* [ ... ] *Il suffit cependant d'avoir corrigé une mauvaise thèse de mathématiques, pour reconnaître combien M. Russell a vu juste. Le candidat se donne souvent beaucoup de mal pour trouver la première équation fausse; mais dès qu'il l'a obtenue, ce n'est plus qu'un jeu pour lui d'accumuler les résultats les plus surprenants, dont quelques–uns même peuvent être exacts.*[140] [79, IV, i]

[139] Some authors even suggest that this goes back as far as Aristotle, which is debatable. From the beginning of the $20^{th}$ century, when binary logic was formalized, a current of logicians exploring the scope of this idea constructed what they referred to as paraconsistent logic (logic that is beyond consistency). This name was introduced in 1976 by the Peruvian philosopher Francisco Miró Quesada (1918).

[140] B. Russell arrives at the conclusion that any false proposition implies all of the other propositions, either true or false. [ ... ] It suffices with having had to grade a bad mathematical thesis to acknowledge that Russell's point of view is accurate. The candidate strives to find the first false equation. But, as soon as they obtain it, it is a simple game of accumulating the most surprising results, some of which may also be accurate.

However, there is reason to think that Poincaré left something important out. We can better see this through an example. The equation 1 = 3 is clearly one of those "false equations" of which nothing seems to come out.[141] But what if we continue with the "error"? By applying the arithmetic rules of whole numbers, we have that 0 = 2, or that 4 = 2 = 0 and so on. These "errors" are referred to as *binary arithmetic* or *base-2 arithmetic*–the technical basis used by computers–and mathematics turns them into solid truths just by making a minor incorporation. They simply write down $1 \equiv 3 \ (mod\ 2)$. The *quodlibet* has become an m-module arithmetic which is indispensable to the study of many aspects of mathematics. We have already used this arithmetic to define dialectic lattices in general. This new structure even allows us to define finite numerical entities that have major implications for various fields of mathematics and science.

From the point of view of the formal properties, it is possible to "demonstrate" this peculiar principle of contradiction. An example of a demonstration can be performed by using the so-called *disjunctive syllogism*: if $(x + y)$ and $Nx$ are theses, then $y$ is a thesis. The difficulty resides in that, in dialectics, the disjunctive syllogism is false. It is very simple to offer a counterexample: $a+0$ is a thesis, $Na$ is one as well, but 0 is not a thesis. Another possible "demonstration" is based on MTE:

| | | |
|---|---|---|
| 1) | $a \,.\, Na$ | starting hypothesis |
| 2) | $a$ | EC in 1) |
| 3) | $Na$ | EC in 1) |
| 4) | $Nb$ | hypothesis |
| 5) | $a$ | reiteration of 2) |
| 6) | $Nb \Rightarrow a$ | conclusion from 4) and 5) |
| 7) | $Na \Rightarrow NNb$ | MTE of 6) |

[141] It is also the statement made by the Christian trinity, put in terms of inventive mathematical language. Birkhoff [4, XII, 6] cites this anecdote by Russell. Russell is reputed to have been challenged to prove that the (false) hypothesis 2 + 2 = 5 implied that he was the Pope. Russell replied as follows: "You admit 2 + 2 = 5; but I can prove 2 + 2 = 4; therefore 5 = 4. Taking away from both sides, we have 3 = 2; taken one more, 2 = 1. But you will admit that I and the Pope are two. Therefore, I and the Pope are one. q. e. d."

| | | |
|---|---|---|
| 8) | $NNb$ | MP of 3) and 7) |
| 9) | $b$ | PNN of 7) |
| 10) | $(a \,.\, Na) \Rightarrow b$ | Conclusion from 1) and 9). |

This demonstration does not tell us anything new and is formally flawed. In the case of a negation–such as $N_0$ in **2Dn** or **3Dn** and presumably in lattices of greater rank–for an atom $a$, we have that $a \Rightarrow a$ and $a \Rightarrow N_1\, a$ given that $N_1\, a = A$. If the previous demonstration *were correct*, we might deduce that $a \Rightarrow b$ must be a consequence of 2) and 9). However, the truth table–see Tables 34, 35 or 36– indicates that this is false. That is, in dialectic logic, we find a counterexample that shows that the formal demonstration is false.

On the other hand, $(a \,.\, Na) \Rightarrow b$, if the negation is strict, does not say anything other than that $0 \Rightarrow x$ is a thesis. This is true for $x = 0, d, 1$ because $f_1 > 0$ and because both $0 \Rightarrow 0$ and $0 \Rightarrow 1$ are theses as well. This point will be analyzed further in all of its complexity, together with the validity of the CI rule.

The demonstration that no (rational) number yields 2 when squared failed to destroy mathematics but rather expanded it through the creation of "irrational" numbers. In much the same way, the impossibility that the square of a (real) squared number could yield –1 led to the creation of "imaginary" numbers. The terminology used in mathematics–both irrational and imaginary–implicitly acknowledges that they stem from contradictions.

Difficulties with a Hamiltonian operator led Paul Dirac (1902, 1984) to conceive of the idea of the "antiparticle" that was discovered shortly after. It also holds on to a name reminiscent of the original contradiction. By the way, it was not the only contradiction that he introduced in science.[142]

These examples show that we must be careful when handling the notion of contradiction, even in exact and formal sciences. In many

[142] Another well-known example is Dirac's $\delta(x)$ "function", something that contravened all the prior definitions of a function. This "function" had already been suggested by Oliver Heaviside (1850, 1925) in 1894 and by Poincaré in 1912. It was subsequently formalized by Laurent Schwartz (1915, 2002) in his theory of distributions.

cases, far from being an obstacle, contradiction has been the source of generation of new knowledge.

## The parallel postulate in geometry

When examining the history of mathematics and science, we can find examples of the application of the dialectic implication function. Let us begin by classic Greek geometry. It is clear that the following has been the sequence of historical events:

1. Towards –500, Thales of Miletus discovered the properties of congruent triangles.

2. Also around –500, Pythagoras discovered the theorem of the hypotenuse of the right triangle.

3. Between –500 and –300, other unidentified mathematicians derived several results linked to the two previous major theorems.

4. Around –350, Aristotle of Megara performed the first formalization of deductive reasoning.

5. Around –300, Euclid discovered the notion of axiom and constructed a deductive theory of geometry and, by extension, mathematics.

6. For 20 centuries, there was doubt about the axiomatic nature of Euclid's statement on parallel line segments.

7. In the $19^{th}$ century, the parallel line segments axiom was proven to be independent from the remaining classical axioms of geometry.

8. Towards mid-$19^{th}$ century and the beginning of the $20^{th}$ century–from Boole to Russell–the theory of logical deduction was formalized.

We will analyze this history from the point of view of the logical validity of the propositions of geometry. Towards –500, the two fundamental theorems had relative logical value; while they were based on the observation of triangles and their properties, they were poorly supported. We might say that their logical value was that of a thesis. A fundamental shift occurred thanks to Euclid's work: axioms, propositions that *were assigned* the logical value *true*, were introduced.[143] With this modification, theorems also became statements with "truthful" value, that is to say, they acquired *the same logical value as the axioms that originated them*.

The fact that the parallel segment axiom posed doubt did not change its logical value, it was also "true"–the only thing was that there was speculation about it being a theorem. In the $19^{th}$ century, the issue of the parallel line was unexpectedly resolved and became a very important element from a dialectic point of view. On the one hand, János Bolyai (1802, 1860) and Nikolai Lobachevsky (1792, 1850) published two separate geometry treatises that posited the existence of more than one parallel line. A couple of decades later, Bernhard Riemann (1826, 1866) presented a geometry without parallel lines. From this moment on, depending on the accepted axiom, three variants of geometry would coexist.[144]

The acceptance of different axioms for parallel lines allows us to build geometries that have *the same logical value as the accepted axiom*. The three versions of the axiom are opposites among themselves but exist simultaneously. This interacts perfectly with Hegel's **D3** lattice. Therefore, for instance, if "there is no parallel line", we can accept that it has *thesis* value, if "there is a single parallel line", that it has *antithesis* value and if "there is more than one parallel line", that it has *synthesis* value.[145] Thus, the scenario for $19^{th}$ century geometry presents itself as perfectly coherent. The theorems of *elliptic geometry* all have value $a$

[143] It is not that axioms are "true" in an epistemological sense, it is just conventional to accept that they are universally valid, as we will see in what follows.

[144] There is at least a fourth variant of geometry, the so-called projective geometry, that does not have parallel lines and is derived from the study of perspective. This geometry can be interpreted in terms of Euclidean geometry by accepting the existence of points, lines or planes to infinity. A coherent geometry was thus obtained, with the peculiarity that the concepts of point and plane were interchangeable.

[145] This allocation of logical values does not correspond to the chronological order of events, but it is more coherent to say that a single parallel line is the synthesis between non-existence and multiple existence.

in **D3**; those of *Euclidean geometry*, value $b$ and those of *hyperbolic geometry*, value $c$. These theorems are contradictory among themselves, but within the framework of dialectic interpretation, they constitute *a single geometry*.

With the discovery of non-Euclidean geometries, the parallel line axiom went on to become a *choice*–it could be either accepted or not, as desired. Once it has been accepted with the value "true", regardless of the statement from the three possible cases–absence of parallels, existence of a single parallel, existence of multiple parallels–, a valid geometry is constructed which proves to be useful in making sense of the universe.

Dialectic implication allows us to understand the existence of deductive theories at once contradictory and valid. Non-Euclidean geometries constitute the paradigmatic case. As has been proposed, the simultaneous existence of the three (or more, see [77, III]) geometries may be analyzed without difficulties in lattice **D3** or higher. Let us consider Euclidean geometry as a deductive system where its statements only contain the subset of the logical values $S_1 = (a, 1)$, a *cone* in **D3**, see Definition 14. We will reserve the values $S_2 = (b, 1)$ and $S_3 = (c, 1)$ both cones in **D3**, for elliptic and hyperbolic geometries.

Further still, in these logical systems, all of the formal properties of implication–including the CI property of implication–are valid, given that, for instance, both $a \,.\, a$ and $a \,.\, 1$ or $1 \,.\, 1$ are theses, as well as $a$ and $1$. Due to the property PM, the principle of mixture, each one of these theories is perfectly coherent on the basis of implication, given that dialectic values do not mix. Additionally, the properties of the logic of predicates are also valid since the properties of classic quantifiers are valid in a cone. In accordance with this, everything happens between dialectic values, just as if it occurred in binary formal logic. This situation offers a major semantic clue as to the use of dialectic logic in science.

The construction of theorems on the basis of axioms–all axioms are worth 1 with the exception of the parallel-line axiom which can be either $a, b, c$ as desired (we will choose the value $a$)– makes theorems have $S_1$ values–the sum of the angles in a triangle, for instance. The logical validity of Euclidean geometry is thus established. Something similar occurs with elliptic geometry and hyperbolic geometry. Changing theorems or demonstrations for each theory is not necessary.

Thus, the construction of theorems can go on without ever encountering a contradiction. All the theorems from the three geometries are simultaneously valid. By extension, we must conclude that both $a \Rightarrow b = 0$ and $a \Rightarrow c = 0$ are contradictions. In this way, each geometric theory can be developed by applying all the formal rules, without noticing that some theorems are worth $a, b$ or $c$ and others are worth 1. This point is crucial as a consequence of the properties of the implication functions.

The example of the three (or more) geometries acts as a *general model* for analyzing the remaining sciences, whether formal, natural or social. It allows us to see why we are able to accept contradictory theories without this implying a violation of the formal rules. To this end, we have made a detailed analysis from a dialectic point of view.

## Dialectics in mathematics

Mathematics has not escaped the general problem of geometry. The development of algebra experienced a similar process. The mathematical concept of “group” was developed in the $19^{th}$ century. What was the connection between group theory and the rest of the “traditional” mathematics?

As proposed by the Euclidean model, theory is, in and of itself, axiomatic as well as deductive. However, we cannot say that the axioms in the theory are “true”. They are actually theses that either apply to the objects at hand, or not. Some sets of objects are groups and others are not. The same occurs with all the algebraic structures introduced from the $19^{th}$ century onwards. Boole’s algebra and lattice theory are included among these structures.

Is “traditional” Aristotelian logic, which was formalized around 1900, true? The answer is no–it is just a set of *theses* that are usually accepted, but that it is not mandatory for us to accept. The existence of a logic more powerful than binary logic is the subject of the present study.

A similar thing happens with the famous result by Kurt Gödel (1906, 1978), which represents a paradigmatic case in the study of the link of the formal theories and the dialectic interpretation of thought. This case is logicians' most convoluted effort to overcome the limitations of binary logic in understanding mathematics and the sufficiently rich formal theories capable of containing arithmetic.

Gödel's result is one of these special cases in which formal chains of arguments are put together that lead to a difficult-to-interpret result. In its original proposition, the procedure is the following, see [29]:

- An arithmetic apparatus is constructed to allow expressing logical and mathematical statements as numbers.
- The existence of arithmetic functions indicating whether a proposition can be proven is demonstrated.
- A (very complex) proposition is constructed, that we will refer to as $G$, whose properties are studied.
- Two propositions *are demonstrated*: $G \Rightarrow \neg G$ and $\neg G \Rightarrow G$.
- This leads either to arithmetic being *inconsistent* or to the existence of propositions that are *non-demonstrable*, such as $G$. This is the issue that we must prove dialectically.

In binary logic, we can resort to the proposition $(\neg p \Rightarrow p) \Rightarrow p$ which has been selected by Frege as an axiom of logic and is recognized as the principle of contradiction, PC–it is argued that, on the basis of this proposition, every proposition is valid. This is not the case with dialectic logic. It is clear that if a proposition can take on *thesis* value and its negation can then take on *antithesis* value, there will be no serious consequences.

Let us then examine Gödel's problem in dialectic terms. Gödel proves the two aforementioned propositions by means of deductive formal chains. Reasoning by the absurd, these two propositions tell us that both $G$ and the *negation* of $G$ are, in some way, true, which indicates that there is an intermediate value between "true" and "false"–what we have referred to as a *dialectic value*. Ultimately, Gödel's theory indicates that *in every axiomatic system* that is ample enough to contain arithmetic, *dialectic propositions can be constructed despite only attempting to fabricate strict truths.* In other words, that it is possible to construct arguments that violate the principle of mixture, PM, something which does no harm at all to dialectic results.

Formally speaking, Gödel's result cannot be symbolized as $1 \Rightarrow a$, where $a$ is a dialectic value, because this is not possible in dialectic implication. It can, however, be symbolized as $N\,G \Rightarrow G$, implying that $G = a$ that is, mathematics allow us to create a *non-decidable* statement that can only be interpreted as a dialectic value.

If we were to proceed as was usual in mathematics throughout history, whenever a contradiction is found, $G$ would be added as a new arithmetic axiom. However, it is possible to speculate that, applied to this new situation, Gödel's own theory would lead to another statement of the type $N\,G_1 \Rightarrow G_1$ and so forth.

Gödel's discovery is not an isolated case in mathematics. There are other cases not as spectacular as this—and they are not even acknowledged as dialectic problems.

Propositions referencing fairly unknown mathematical problems pose a similar problem. As an example, let us think of Christian Golbach's (1690, 1794) conjecture, formulated in 1742–every even number is the sum of two prime numbers–or the simple affirmation that in the decimal development of $\pi$ for instance, the number 8 exists 100 times in a row. Based on a little known proposition, very interesting speculations can be made which are within the scope of dialectics.

It is worth illustrating these problems through a simple mathematical example. Let us consider the following classic problem: proving that an irrational number to the power of another irrational number can yield a rational result. There is a demonstration–one not accepted by *constructive mathematicians*–that falls within the scope of dialectics. We will consider its propositions:

- *p*: there are two irrational numbers, $x, y$ that meet $x^y$ being rational.
- *q*: $a^a$ is a rational number, where $a = \sqrt{2}$.

For the time being, we will ignore the logical value of the propositions

$p$ and $q$. It follows immediately that $q \Rightarrow p$ is a thesis given that if $q$ is a thesis, the theorem that we are trying to prove is also a thesis. But $N\,q \Rightarrow p$ is also a thesis, since if $a^a$ were irrational, then $(a^a)^a = a^2 = 2$ and two irrational numbers can also be found under the requested conditions. From $q \Rightarrow p$, $N\,q \Rightarrow p$ being theses, it follows that $p$ is a thesis.

Spontaneous reasoning tells us that: either $q$ holds true and then $p$ is true, or $Nq$ holds true and $p$ is also true. However, it is not as simple to deduce this result from the formal rules. A possible argument would be the following: 1) $q \Rightarrow p$; 2) $N\,q \Rightarrow p$; 3) $N\,p \Rightarrow NN\,q$ by MTE in 2); 4) $N\,p \Rightarrow q$ by PDN in 3);[146] 5) $N\,p \Rightarrow p$ by T in 1) and 4); 6) $NN\,p$ by PC in 5); 7) $p$ by PDN in 6). This reasoning shows that, from a formal standpoint, it is not necessary to assume that the alternatives to $q$ are either true or false. It also proves that $p$ is a thesis in dialectics because it follows the formal rules of implication. We cannot prove that the theorem is *true*; rather, than it is a thesis in a dialectic sense. Deep down, the application of an argument by the rules yields a result that is weaker than usual in mathematics. In this sense, constructivist mathematicians are on to something. They are wrong, however, in disputing the validity of the theorem.

As in the last case, let us consider the conjecture made by Pierre de Fermat (1607, 1665): there is no solution to the equation $x^n + y^n = z^n$ for $n > 2$, where $x, y, z$ are whole numbers. The first demonstration of this conjecture did not occur until 1994 and was performed by Andrew Wiles (1953). With an extension of over 150 pages, it delved into very diverse areas of mathematics. For 358 years, the nature of this conjecture was unknown. Even today, in light of its complexity, it is worth wondering whether a demonstration outside of the scope of the natural numbers, where it has been proposed, is possible.

These results illustrate the new possibilities for analysis that dialectics have to offer to some of the classic mathematical problems. In other

[146] This step is not as immediate according to the formal rules. It goes as follows: 3) $N\,p \Rightarrow NN\,q$ by MTE in 2); 3a) $N\,p$ as starting hypotheses for a subordinate reasoning; 3b) $N\,p \Rightarrow NN\,q$ as a copy of 3) in the subordinate reasoning. 3c) $NN\,q$ by MP in 3a) and 3b); 3d) $q$ by PDN in 3c); 4) $N\,p \Rightarrow q$ by the introduction of the implication and end of the subordinate scheme.

words, mathematics demands a logic that is more intricate than binary logic. It is possible that only dialectics can understand the mathematics of the present, given the leap of quality that has taken place.[147]

## Dialectics in information science

Information science displays several cases that possibly require dialectic treatment. Without meaning to present an exhaustive list, we will mention the following:

- Halting's theorem,
- the double definition of real numbers,
- NP-complete problems.

Alan Turing's (1912, 1954) Halting theorem establishes the scope of action of a machine that handles symbols. This theorem is based on the construction of a contradiction and, therefore, one may obtain different conclusions when it is subjected to dialectic analysis.

In mathematics, there is a double definition for real numbers. On the one hand, we have the classical definition made by Richard Dedekind (1831, 1916), using cuts[148] and on the other, the definition used by Georg Cantor (1845, 1918), as an infinite succession of decimal digits after the decimal point. The equivalence between these two definitions is more than a little dubious and a dialectic problem possibly exists here.

The computational complexity of the algorithms that depend on a parameter $n$ allows us to sort them into two groups: those where their calculus complexity increases as a polynomial in $n$–for example, calculating the $n$ decimal figures of the square root of an integer–and

[147] José Luis Massera [64, 65] showed that the notion of "rigor" in mathematics is one that has changed throughout history. This is where he supports his entire defense of the dialectic quality of mathematics. I think the problem goes beyond this idea as proposed, which is clearly true.

[148] Cuts are a classification of rational numbers in two classes. The weak spot of this definition–evidenced by Turing's work–is that a *precise procedure* is necessary–that is, an algorithm–in order to know whether a number belongs to one class or the other. This point of view makes Turing's *computable numbers* the only actual real numbers.

problems which are more complex and whose calculus increases in a Non-Polynomial (NP) manner with $n$–for instance, finding the optimal path between two points in a network of roads having $n$ intersections. Known NP problems are equivalent among themselves, but they have not really been proven to be non-polynomial. A problem leading to a dialectic proposition may also exist here.

## Introduction to dialectics in the natural sciences

The natural sciences are, by their very epistemological nature, *experimental*.[149] At a certain point in their development, these sciences end up admitting a formulation similar to that of mathematics: a deductive, essentially quantitative theory is constructed on the basis of a few principles. This does not change its experimental quality. The analytical expose is merely a way of presenting the results. At all times, an experiment or an observation can controvert these theories and call for a complete revision of the results.

Henri Poincaré, with his unique insight on the philosophy of science, would say:

> *Les Anglais enseignent la mécanique comme une science expérimentale ; sur le continent, on l'expose toujours plus ou moins comme une science déductive et* a priori. *Ce sont les Anglais qui ont raison, cela va sans dire* [ . . . ][150] [77, VI]

The double exposure admitted by the natural sciences, in their ad-

[149] This must be understood in a broad sense. For example, neither geology nor astronomy or history can truly carry out experiments. But they can perform experimental observations, take measurements, and to a lesser extent, perform some experiments. Measuring the speed of debris in a landslide or the amount of salt contributed by a river, sending space probes to take pictures, collect samples or analyze celestial bodies are–in a way–experiments. The formation of the USSR, cooperatives and other similar cases may be considered forms of social or historical experimentation, as in the past other experiments have taken place, such as, for example, Akhenaten's religious reform in pharaonic Egypt. In this sense, history can also be considered (somewhat) experimental.

[150] The British teach mechanics as an experimental science; in continental Europe, it is more or less presented as a deductive, *a priori* science. Needless to say, the British are right [ . . . ].

vanced state, is proposed here. Beyond Poincaré's preferences–which one may certainly agree with–it is worth asking, how do these two ways of presenting the experimental sciences differ from one another?

The answer is far from trivial and is one of the topics analyzed in this chapter. As an introduction to the subject, we can resort to Newton's analysis on gravitation. The duality of criteria pointed out by Poincaré–as occurs with all the sciences that attain training-level–holds true in this case: the possibility of formulating them in an *argumentative* manner based on observational or experimental results or the possibility of formulating them in an *axiomatic* manner based on a reduced set of equations that double as axioms.

Page 166 presents the equation for the experimental argumentation of gravitation. Conversely, based on three "axioms"–the two laws of motion, see page 237 (Mov.) and the law of gravitation (G)–the experimental laws of Galilei, Kepler and the observation of the Flamsteed comet can be deduced.

In conclusion, Poincaré is right about the fact that there are two ways of formulating mechanics; he is not so in saying that one of the formulations is preferable to the other.[151] As we will see in what follows, this situation can be further generalized to all the branches of science.

## Introduction to relationships between physical theories

In [44, IV] Werner Heisenberg (1901, 1976) classified physical theories–in the state they were in at the middle of the $20^{th}$ century–into four different branches:

- F1 Newton's mechanics.
- F2 Thermodynamics and statistical mechanics.
- F3 Electricity, magnetism, field theory, relativity.
- F4 Quantum physics.

[151] It is possible for mistrust in formalizations to have been originated in his mistrust in the formalization of binary logic. If this presentation of dialectics is accurate, Poincaré was also right about this.

He then established the following relations of dependence. F1 $\subset$ F3 if $c$–the speed of electromagnetic waves–is infinite, F1 $\subset$ F4 if $h$–Planck's constant–is negligible. He fails to establish a relation for F2. Finally, there is the question of whether a theory F exists–forever referred to as the "unified theory"–that would comprise all the branches of physics (and chemistry). This aspiration–further made complex by the successive discoveries in F4–still has a place in the collective imagination of physicists.

In order to analyze these problems, we will study the paradigmatic case of mechanics and the branches of physics associated with it. Mechanics, which are preoccupied with matter and its movement, are at the core of physics and chemistry. For this reason, it constitutes a good example when analyzing the role of dialectic logic in scientific structures. It is reasonable to assume that the rest of the sciences, as they become quantitative and allow for deductive formulation, will encounter similar structural problems.

## $19^{th}$ century mechanics

Towards the end of the $17^{th}$ century, Newton presented mechanics in an axiomatic manner. He began with two basic definitions and an appendix:

- *Quantitas Materiæ est mensura ejusdem orta ex illius Densitate et Magnitudine conjunctim.*[152] [67, 68, I, *Definitiones*, i].
- *Quantitas motus est mensura ejusdem orta ex Velocitate et quantitate Materiæ conjunctim.*[153] [67, 68, I, *Definitiones*, ii].
- *Tempus absolutum verum et Mathematicum* [ ... ] *Spatium absolutum natura sua absq; relatione ad externum quodvis semper manet similare et inmmobile* [ ... ][154] [67, 68, I, *Definitiones, Scholium*].

[152] The quantity of *matter* is its measure, as arising from density and volume conjointly.
[153] The quantity of *motion* is its measure, as arising from the velocity and quantity of matter conjointly.
[154] Absolute, true, and mathematical *time*, in itself and by its own nature, flows equably without relation to anything external [ ... ] Absolute *space*, by its own nature and without reference to anything external, remains always similar and immovable [ ... ]

There are two laws of movement–Newton formulates three, but the first is contained in the second–which are:

- *Mutationem motus propotionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*[155] [67, 68, I, *Axiomata sirve leges motus*, ii].
- *Actioni contrariam semper et æquales esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.* [156] [67, 68, I, *Axiomata sive leges motus*, iii].[157]

The following classic statement results from the laws of motion:

$$\frac{d}{dt}(m\,\vec{v}) = \vec{F}$$

allowing us to analyze projectiles, planets and even variable-mass systems such as spaceships. Based on this famous equation, the $18^{th}$ and $19^{th}$ centuries saw a great advancement in the knowledge of the movement of matter (Matt. as the abbreviation) and in a new axiomatic formulation.

Two new formulations for the theory of motion are constructed in the $19^{th}$ century: the equations of Joseph-Louis Lagrange (1736, 1813) and the equations of William R. Hamilton (1805, 1865). These equations were of a less general nature than Newton's statement, but they were nonetheless decisive to $20^{th}$ century mechanics.

The exposition of Lagrange's mechanics is based on the function $L(q, \dot{q}, t)$ –referred to as Lagrange's function–where $q, q, t$ are, respectively, the coordinates, the derivatives with regards to time of the coordinates of the material points of a system, and time. The function complied with the following axioms:[158]

---

[155] A *change in motion* is proportional to the motive force impressed and is made in the direction of the straight line in which that force is impressed.

[156] To every *action* there is always opposed an equal *reaction*: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

[157] It is hard not to see in the law of action and reaction a dialectic statement of unity and struggle of opposites, something which confirms the dialectic nature of the *Principia*.

[158] There are many formulations of this mechanic. Among them, it is preferable to

1. If a system is formed by two sub-systems, $A, B$, with no interactions between themselves, then, Lagrange's function for the total system is $L = L_A + L_B$.

2. The motion of the system between $q_1$ and $q_s$ makes the *action* integral $\int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$ minimal.

3. Lagrange's function of a system of points interacting among themselves is given by $L = \frac{1}{2} \sum a_{ij}(q)\, \dot{q}_i\, \dot{q}_j - U(\vec{r_1}, \vec{r_2}, \cdots, t)$ where $\vec{r_i}$ is the position vector for point $i$.

4. The basic system of reference for mechanics–*Galilei's principle of relativity*–is homogeneous in space and time.[159]

The essential difference between Newton's and Lagrange's mechanics—and other theories derived from these—lies in the independence of the interaction $U$ with the speeds $\dot{q}_i$. This occurs with friction in air or in a liquid, for example.[160] It also occurs in the case of magnetic forces.

Another way of writing the motion equations for the systems derived from Lagrange's equations: Hamilton's equations. This is important to the development of quantum mechanics and can be found in [51, VII, 40]. In essence, it consists in a change of variables, in which the generalized speeds $\dot{q}$ are replaced by the generalized impulses $p$ and Lagrange's function is replaced by Hamilton's $H$ function, defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}} \qquad H = \sum p_i\, \dot{q}_i - L.$$

follow Lev Landau's exposition in his series on theoretical physics, written together with Evgeny Lifchitz [51]. This exposition, aside from being axiomatic in nature, has been authored by a Nobel prize-winner and a materialistic physicist, two conditions that stand behind its selection.

159 Strictly speaking, this principle was defined by Newton and is described in [67, 68, I, Definitiones, Scolium].

160 Landau would even say: *le problème du movement d'un corps dans un milieu n'est plus una problème de Mécanique* (the problem of the motion of a body in a [material] medium is not a problem of mechanics) [51, V, 25].

$H$ is the energy of the system, as can be easily proven. The two canonical equations come as a result of this change of variables:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

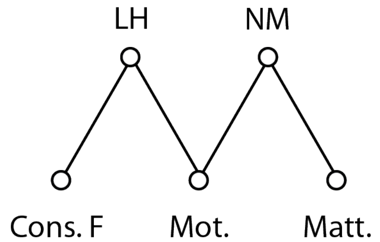


Figure 30: Argumentative diagram in $19^{th}$ century mechanics.

The logical relations between the different formulations of mechanics are presented in Figures 30 and 31. In its *argumentative* form, Newton's mechanics (NM) are based on the laws of motion (Mot.) and the properties of matter (Matt.). Conversely, Lagrange-Hamilton's equations (LH) are based on the existence of conservative forces (Cons. F.) or those that derive from a potential. They fail to study the general case, where forces may be dependent on speed or dissipate energy.

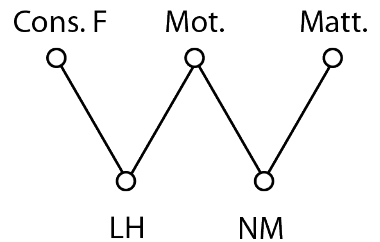


Figure 31: Axiomatic diagram in $19^{th}$ century mechanics.

The logical relations immediately follow. In the fragment of the lattice in Figure 30, the order relation means a greater logical value or a greater explicative value. At the same time, the only thing in common between LH and NM are Newton's motion equations (Mot.). An observation follows immediately. It has been depicted–only a fragment of a lattice–that both LH and NM are of a higher logical value than Cons. F., Mot. and Matt. It seems only natural that this is so.

This representation has an immediate dialectic interpretation, for example in **2Dn**. If we were to consider the logical values Cons. F., Mot., Matt., LH, NM, 1, then in the cone $S_1 = (Cons.F., LH, \ldots, 1)$ Lagrange-Hamilton's theory can be argued–as in the case of non-Euclidean geometries. Also in the cone $S_2 = (Mot., Matt., NM, \ldots, 1)$ Newton's mechanics can be argued on the basis of the "axioms" Cons. F., Mot. and Matt. The necessary mathematical theorems have logical value 1 and are accepted as absolute truths. *In a dual manner*, by reversing the figure,[161] the theories become based on the "axioms" LH and NM and from these, by applying all the logical formalism, the laws of matter, motion and conservative forces are proven. The ellipsis in the definition of the cones allows for the possibility of yet-to-be-developed theories with a higher logical value.

What are the advantages of a dialectic formulation? Several. To begin with, it establishes a hierarchy in the logical levels of each area of knowledge. Secondly, by failing to include the value 1 in the lattice fragments[162] it is clear that no theory intends to be absolutely true, something which leaves the door open to expanding our knowledge towards higher logical levels, as we will see in what follows.

## 20$^{th}$ century mechanics

Mechanics underwent two major revolutions in the beginning of the 20$^{th}$ century: relativistic mechanics and quantum mechanics. These two branches of physics derive from 19$^{th}$ century mechanics, electromagnetism and the knowledge of the structure of matter.

Relativistic mechanics originated due to an incompatibility between relative motion and electromagnetism (EM).[163] In James Clerk Max-

[161] The reason for considering the reverse lattice can be found, essentially, in the laws of motion, Mot., which come as a consequence of both LH and NM, two "opposing" theories.

[162] This is not strictly true. Mathematical theorems with value 1 are part of formal developments, as is universally accepted. In reality, they are implicit in partial lattice diagrams.

[163] Let us consider a homogeneous electric charge distributed according to an indefi-

well's (1831, 1879) equations, the speed of electromagnetic waves is a universal constant–something which was proven experimentally by Edward Morley (1838, 1923) and Albert Michelson (1852, 1931) in 1887–against Galilei's and Newton's composition of speeds.

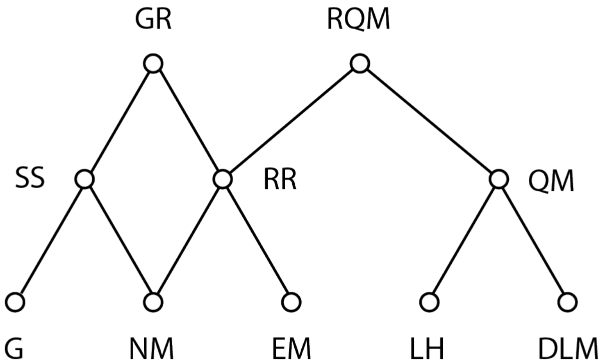

Figure 32: Argumentative diagram in $20^{th}$ century mechanics.

In 1095, Albert Einstein (1879, 1955) [84] proposed a new transformation equation for relative motion that accounted for the constancy of the speed of propagation of electromagnetic waves. These equations, known as restricted relativity (RR) resolved all the issues, see Figure 32.[164]

In this fundamental work, Einstein offered a new way of interpreting the transformation equations that Hendrik Lorentz (1853, 1928), George FitzGerald (1851, 1901) and also Henri Poincaré had already discovered. In 1908, Hermann Minkowski (1864, 1909) interpreted–as Poincaré had already anticipated–the transformation equations as a four-dimensional space, with time being the fourth imaginary dimension (in a mathematical sense), with an Euclidean metric. This did

nite straight line. A resting observer with regards to the line will only detect an electric field. Aside from the electric field, an observer who is moving at a constant speed parallel to the charged line, will also observe a magnetic field, given that the charge–he is observing while in motion–forms an electric current that creates this field. Ultimately, Newton's principle of relativity does not hold true. For this reason, Einstein's work is entitled *Elektrodynamik bewegter Körper* (On the Electrodynamics of Moving Bodies).

[164] No direct link between EM and LH appears. Although they are compatible and can be harmonized between each other, they fail to generate any higher level theories worthy of mention.

away with Newton's idea of absolute space and time, interdependent between each other.[165]

Shortly after, in 1916 [84], Einstein would generalize the principle of relativity and offer a new interpretation for planetary motion (SS), by means of the so-called general relativity (GR). The basic idea behind it is that matter curves space and forces bodies to move according to the minimal trajectory (geodesic): mass determines the curvature of space, whereas the curvature determines the motion of matter.[166] This way of presenting gravitation resolves the issue of the equivalence between the inertia of matter and the gravitational pull exercised by it.

Figure 32 shows that Newton's mechanics (NM), together with the solar origin of gravitation–as proven by Newton, see page 30–(G), give way to the Solar System (SS) theory. Electromagnetism (EM) and Newton's mechanics (NM) generate restricted relativity (RR). The motion of the solar system (SS) and restricted relativity (RR) give way to general relativity (GR).

The observation of matter during the $19^{th}$ century and the beginning of the $20^{th}$ century–especially Dalton's, Lavoisier's and Mendeleev's (DLM) chemistry–yielded a number of new results which ultimately led to quantum mechanics (QM) and subsequently, relativistic quantum mechanics (RQM). The following is a most certainly incomplete list of these:

- *Dalton's Law* (1805). Matter is made up of atoms grouped into molecules that configure different structures and combinations.[167]
- *Dalton's Law* (1805, etc.). An "atomic weight"–mass would be more accurate–can be associated with each kind of atom, connected to the manner in which molecules are formed. A relative measure is derived to hydrogen, which is taken as the unit.

[165] In this metric, space and time "intermingle", such as an electric field "intermingles" with the magnetic field. One cannot but see the unity and struggle of opposites: space-time, electric field-magnetic field.
[166] Once again, we find the existence of two synchronic opposites: matter and the curvature of space. Each one exists because of the other.
[167] A result considered by Feynman, at the start of his physics, as an immense achievement.

- *Mendeleev's Law* (1865–1870). If the different known atoms are sorted by their atomic weight, one can see there is a certain periodicity in their chemical and physical properties and eventually some "gaps" are detected.
- *Bunsen-Kirchhoff Law* (1860, etc.). etc.) To each atom corresponds a spectrum–that is, a discrete set of frequencies–of light that is issued or absorbed when excited under certain conditions.
- *Mendeleev's Law* (1865–1870). All of the "gaps" were filled by elements which had not yet been discovered.[168]
- *Law of Thomson, Rutherford and others* (1896–1914). Particles smaller than the atom exist. Atoms are a complex structure made up of these particles.

The structure of the atom was the main topic of study at the beginning of the $20^{th}$ century. Subatomic particles did not behave as macroscopic particles did. The electrons that orbited the atomic nucleus did not emit any energy and the spectra of energy emitted by the atoms were not continuous. Thus began a series of discoveries that revealed various aspects of the structure of matter:

- *Planck's Law* (1900). The emissions from a hot body can be explained by oscillators with discrete energy, in multiples of $h\,f$, where $h$ es is Planck's constant, $f$ is the frequency emitted.
- *Einstein's Law* (1905). The emission of electrons by the incident light–the photoelectric effect–is explained by Planck's law. The emission was produced if $h\,f \geq E$ where $h$ is Planck's constant, $f$ is the frequency of the incident photon and $E$ is the energy necessary to release one electron from the material.
- *Electron diffraction* (1924–1927). Electrons behave as if they were waves that comply with the hypothetical matter waves proposed by Louis de Broglie (1892, 1987).[169]

[168] The case of *ekasilicon*–currently called germanium–was the first. The series of lanthanides, actinides and transuranic elements have been the most recent.

[169] It is hard to imagine a physical law that can better respond to the dialectic idea of unity and struggle of opposites. Wave and particle are, without a doubt, physical

The confluence of the idea of the periodic table of elements, the discrete spectra of the atoms and energy behaving in a discrete manner, led, in 1913, to Niels Bohr's (1885, 1962) formulation of the atom: electrons can only occupy certain orbits, each one with a specific amount of energy. The passage of one electron from an orbit with a greater amount of energy, $E_2$, to one of lesser energy, $E_1$ releases a photon with energy $E_2 - E_1 = h\,f$. In a way, the process is the reverse of the photoelectric effect: a photon releases an electron, an electron releases a photon.

De Broglie's hypothesis allows us to intuitively explain Bohr's discrete orbits: a whole number of wavelengths must be able to fit into the orbit, which means that only certain orbits are allowed to the electrons. This begins to explain the periodic table of elements.

With these elements, in 1925, Heisenberg published a revolutionary work [44].[170] Only analyzing observable physical parameters was proposed: the energy of the electron was observable, the details of the orbit and its motion were not. The work, then, revolved around expressions oriented at rebuilding the spectrum of hydrogen, but not at trying to build a dynamics of the electron. The idea was revolutionary– but the exposition was quite cryptic, as appointed by [1]– and garnered importance with the publication of a work by Max Born (1882, 1970) and Pascual Jordan (1902, 1980) [24]. There, the use of matrix calculus was introduced in physics and the results proposed by Heisenberg were put in understandable terms. Additional notes included a fundamental result for quantum mechanics which would establish a bridge with Hamilton's mechanics:

$$p\,q - q\,p = \frac{h}{2\pi i}\,I$$

where $p$ and $q$ the matrixes of momentum and position, $I$ is the identity matrix. In 1926, Wolfgang Pauli (1900, 1958) calculated, by means of the new formalism, the spectrum of hydrogen, thus formalized Bohr's atom and discovered the electron's *spin*.

objects whose properties are different and opposed to each other. The hypothesis establishes that every particle with a momentum $p$ displays an undulatory behavior with a wavelength of $\lambda = h/p$, where $h$ is Planck's constant.

[170] Simultaneously, Paul Dirac, see [18], was working with the same idea as Heisenberg.

Erwin Schrödinger (1887, 1961), unlike Heisenberg and others, followed the path of De Broglie's matter waves, see [89].

> The chief advantages of the present wave–theory are the following. a. The laws of motion and the quantum conditions are deduced simultaneous from one simple Hamiltonian principle. b. The discrepancy hitherto existing in quantum theory between the frequency of motion and the frequency of emission disappears in so far as the latter frequencies coincide with the differences of the former. [ ... ] c. It seems possible by the new theory to pursue in all detail the so–called "transitions", which up to date have been wholly mysterious. d. There are several instances of disagreement between the new theory and the older one as to the particular values of the energy of frequency levels. In these cases it is the new theory that is better supported by experiment. [89, #1]

Schrödinger applied the principle of least action–the Huygens-Fermat principle, which establishes that electromagnetic waves follow a minimal path–to the matter waves proposed by De Broglie. He then observed that the resulting equations had a similarity with the Hamiltonian operator of classical mechanics.

> Take this function [the classical Hamiltonian] to be a homogeneous quadratic function of the momenta $p_x^2$ etc. and of unity and replace therein $p_x, p_y, p_z$ by $(h/2\pi)(\partial\psi/\partial x)$, $(h/2\pi)(\partial\psi/\partial y)$, $(h/2\pi)(\partial\psi/\partial x)$, $\psi$ respectively. There results the integrand of (20) [the condition of least action]. This immediately suggests extending our variation problem an hereby our wave–equation (16) to a wholly arbitrary conservative mechanical system. [89, #7]

From this observation arose the so-called Schrödinger's equation, which allowed for the development of the entirety of quantum mechanics:

$$H\,\psi(t) = \frac{ih}{2\pi}\frac{\partial\,\psi(t)}{\partial t}$$

where $H$ is the Hamiltonian operator in which the classical momentum $p_x$ are replaced by the operators $(h/2\pi)(\partial\psi/\partial x)$, etc.[171]

Subsequently, it was proven that Heisenberg's, Born's and Jordan's formulation was equivalent to that of Schrödinger, see, for example, [18]. Quantum mechanics (QM) was thus consolidated, see Figure 32. The undulatory formulation presented the difficulty of interpreting the physical meaning of the so-called "wave function" $\psi$. In general, it is accepted that its normalized module–given that it is a complex variable function–is the distribution of probabilities at the particle's position.

The wave function allowed for the definition of what we know as the *principle of uncertainty*, as advanced in Heisenberg's first work. This principle establishes that it is not possible to precisely know a particle's position and impulse. In mathematical terms, it is expressed as $\Delta x\,\Delta p_x \geq h/4\pi$ where $\Delta x$ and $\Delta p_x$ are the typical deviation of the measures of the position and momentum at coordinate $x$.[172]

The interpretation of the wave function led to a philosophical discussion that can be summarized by Einstein's famous quote:

> Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the "old one". I, at any rate, am convinced that He is not playing at dice. [Einstein's letter to Born, 4–dec–1926]

Aside from Einstein's epistemological objection, there was still a more serious issue to resolve. Schrödinger's equations were second-degree differential equations in space, but they were first-degree in

[171] The formalism replaces the momentum by $i\,h/2\pi\nabla$ where $\nabla$ is the gradient operator and the Hamiltonian operator is $H = p^2/2m + U$, where $U$ is the potential energy. The total energy is replaced by the operator $(ih/2\pi)(\partial/\partial t)$.

[172] Two variables whose operators are not commutative between themselves obey to an equation of indetermination. They are, from a dialectic standpoint, opposing variables subjected to a quantitative restriction.

time, contradicting the theory of relativity, in which time and space were "intertwined" and were two aspects of the same phenomenon.

In 1926, Oskar Klein (1894, 1977) and Walter Gordon (1893, 1939) proposed an equation that resolved this issue and used Schrödinger's technique of extending the classical formalism by means of restricted relativity. Subsequently, in 1928, Dirac proposed another equation with additional consequences.[173]

We will consider Klein-Gordon's equation as an example of the formalism used. In restricted relativity, the total energy of a particle had the following expression:

$$E = \sqrt{p^2\, c^2 + m^2\, c^4}$$

where $E$ is the total energy, $p$ is the momentum responsible for kinetic energy, $m$ is the mass responsible for the energy at rest and $c$ is the speed of the electromagnetic waves. Naturally, this equation that features square roots does not allow for the direct application of Schrödinger's formalism. Klein-Gordon's solution was to square the entire equation, applying the formalism in order to obtain:

$$\left( -\left( \frac{ih}{2\pi} \nabla \right)^2 + m^2\, c^4 \right) \psi = \left( \frac{ih}{2\pi} \frac{\partial}{\partial t} \right)^2 \psi.$$

Dirac sought to find a more elaborate way of doing away with the square root, which led to the introduction of the spin and also antiparticles. This is how relativistic quantum mechanics (RQM) came to be.

The dialectic interpretation of all these theories call for a **3Dn** lattice[174]–or one of higher complexity–in which the following logical values can be associated: G, NM, EM, LH, LDM, SS, RR, QM, GR, RQM.[175]

[173] Dirac's argument is the following: "There is no need to make the theory conform to general relativity, since general relativity is required only when one is dealing with gravitation, and gravitational forces are quite unimportant in atomic phenomena." [18, XI, 66]

[174] Strictly speaking, if we wish to include the results from Figure 32, we must work with the **4Dn** lattice. The lowest logical level has only been omitted for purposes of simplifying the diagram.

[175] Due to an abuse of the language, the same symbol is used to refer to a theory and its corresponding logical value.

Analogously to the previous section, in $S_1 = (G, SS, RG, \ldots, 1)$–a cone in **3Dn**, for instance–an argument in favor of gravitation (G), Newton's mechanics (NM), electromagnetism (EM) as basic theories, the solar system (SS) and restricted relativity (RR) as theories of a higher logical level, and finally, at the highest logical level, general relativity (GR).[176] This does not preclude the existence of a theory of an even greater logical level, like the much-sought "unified field". Analogously, in $S_2 = (LH, MQ, MQR, \ldots, 1)$ quantum mechanics (QM) and relativistic quantum mechanics (RQM) can also be argued.

In a more general manner, $S_0$ = (MN, SS, RR, RG, MRQ, …, 1) –a cone in **3Dn**, for instance–can be defined, where almost everything can be made compatible. In much the same way, $S'_0$ = (MM, RR, RG, MRQ, …, 1) can be considered to be made compatible the theory.

As in the previous case–by applying the duality shown by Poincaré–we can synthesize the diagram and thus build an axiomatic version. In this case, the axioms are two: general relativity (GR) and statistical quantum mechanics (SQM). When considering regions foreign to the subject matter, GR becomes restricted relativity (RR); when considering small speeds of movement—as compared to $c$, the speed of electromagnetic waves–we obtain Newton's mechanics (NM), Maxwell's classical electromagnetism (ME) and the Newtonian theory of gravitation (G).

In a similar manner, for small speeds of movement, SQM becomes quantum mechanics (QM), from which it can be deduced–when Planck's constant, $h$, is disregarded–in Hamilton's equations and, in consequence, Lagrange's as well. QM also explains the existence of the different chemical elements and the formation of molecules (LDM).

Subsequent studies–when discovering new elementary particles–increased the complexity of SQM but their consideration is outside of the scope of this book. The only purpose was that of applying dialectics to current physics.

[176] It is worth noting that general relativity employs the idea that the Sun is responsible for planetary motion and also use the value of the Cavendish gravitational constant.

## Statistical mechanics

Statistical mechanics were discovered by Ludwig Boltzmann (1844, 1906) for Newtonian material point systems and later, in the $20^{th}$ century, it expanded to quantum particles. The classical example is a gas where each molecule moves with a certain speed within a closed container, but collides with the walls of the container or with other molecules.[177] We are interested in studying the properties of this balanced system, a quality which is attained precisely due to the collisions between molecules or of these with the walls.

An isolated system formed by "many particles", with a total energy and a certain number of particles–both constant–has many possible internal *micro-states*.[178] In its balanced state, all the micro-states are supposed to be equally probable. The properties of the system that can be measured are *new properties* of the entire system and not of its particles; see Landau's comment on page 56. The speed or the energy of each particle cannot be known, but their average values can. Systems of this nature have new observable properties, the most characteristic of which is *temperature*. Based on energy, temperature and other system parameters, we can measure other variables of interest.

The fundamental concept in statistical mechanics is the so-called *partition function*. In a discrete system, where the total energy $E_i$ of each micro-state, the partition function $Z$, is defined as:

$$Z = \sum_i e^{-\frac{E_i}{k\,T}}$$

where $i$ is the index of each micro-state, $k$ is Boltzmann's constant and $T$ is the absolute temperature of the system. In a continuum of micro-states, the sum becomes an integral. Based on the partition function, the remaining observable variables for the balanced system are defined.

[177] It is worth remembering that although the basic ideas behind statistical mechanics were developed in the second half of the $19^{th}$ century, most German physicists did not believe in the reality of atoms and molecules, such as Ernst Mach, Wilhelm Ostwald or Max Planck. Planck gives a clear testimony of this attitude. He was one of the founding fathers of quantum mechanics, despite himself.

[178] A micro-state consists in the position and speed or momentum of each one of the particles. Saying "many particles" means, for example, more than $10^{20}$ particles.

There are three major cases where statistical mechanics are applied: ideal gases and two complementary–or opposing–cases of elementary particle systems: *bosons* and *fermions*. Fermions are particles that meet Pauli's exclusion principle: two particles in the same state cannot exist in a micro-state. Conversely, bosons fail to meet the principle and any number of particles may exist in every possible state.

These considerations have given way to three major statistics: classical Maxwell-Boltzmann (MB) statistics, Bose-Einstein (BE) fermion statistics and Fermi-Dirac (FD) fermion statistics.

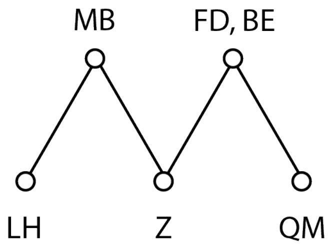


Figure 33: Logical relations in statistical mechanics.

Figure 33 represents the relations between Lagrange-Hamilton (LH) mechanics, quantum mechanics (QM) and Boltzmann's (Z) notion of micro-state and partition function.

## The dialectics of social classes

The dialectics of social classes were first established in the *Communist Manifesto*.

> *Freier und Sklave, Patrizier und Plebejer, Baron und Leibeigener, Zunftbürger und Gesell, kurz, Unterdrücker und Unterdrückte standen in stetem Gegensatz zueinander* [ . . . ] *Im alten Rom haben wir Patrizier, Ritter, Plebejer, Sklaven; im Mittelalter Feudalherren, Vasallen, Zunftbürger, Gesellen, Leibeigene,* [ . . . ] *Aus den Leibeigenen des Mittelalters gingen die Pfahlbürger der ersten Städte hervor; aus dieser Pfahlbürgerschaft entwickelten sich die ersten Elemente der Bourgeoisie.*[179] [61, I, 1-6]

[179] Freeman and slave, patrician and plebeian, lord and serf, guild-master and journey-

Social classes may be multiple synchronic opposites. In capitalism, we find the following classes: bourgeoisie, proletariat, peasants and the *middle classes*.[180] They are synchronic opposites. The following is the dynamic between them: peasants can become proletariats, sometimes middle classes; proletariats can become middle classes or bourgeoisie; the middle classes can go on to be proletariat or bourgeoisie, rarely peasants; the bourgeoisie can go on to become part of the middle classes and sometimes the proletariat. We are also referring to middle classes in plural–a more thorough analysis may also distinguish between opposing middle classes: intellectuals, liberal professionals, officials, etc., with differing interests between classes.

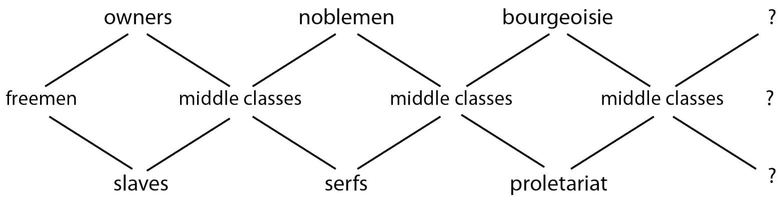


Figure 34: Lattice of materialistic history in Europe.

Figure 34 offers a simplified version of the history of Europe.[181] The diagram shows the different synchronic opposites–owners and slaves, noblemen and serfs, bourgeoisie and salaried employees–and also the becoming of pairs of opposites. It also suggests something which is ac-

man, in a word, oppressor and oppressed, stood in constant opposition to one another [ ... ] In ancient Rome we have patricians, knights, plebeians, slaves; in the Middle Ages, feudal lords, vassals, guild-masters, journeymen, apprentices, serfs, [ ... ] From the serfs of the Middle Ages sprang the chartered burghers of the earliest towns. From these burgesses the first elements of the bourgeoisie were developed.

[180] In contemporary times, sociologists have dubbed *middle class* to what is referred in Spanish as the materialistic *estamentos*. This happened because the natural languages did not have–as the Spanish does–an accurate way of designating these. However, low Latin has the word *stamentum* to refer to the members of urban commercial corporations and this is its precise meaning. In other European languages, the Latin word was not adopted and something derived from the Germanic languages, *burg*–from which bourgeoisie comes from, the middle class in medieval cities–was preferred, a term that first designated the fortresses and then the burgesses.

[181] Without a doubt, the case with the most stages: slavery, feudalism and capitalism.

cepted but *also controversial*:[182] that the free middle classes will give way to the new dominant classes.[183] It also suggests that the free middle classes of capitalism will be in charge of building a new society or become the end of history.[184]

The importance of the middle classes is evident in the existence of political movements aimed at transferring social power to the dominant and the dominated classes, at the expense of eliminating the middle classes. These movements go by very different names and range from the authoritarianism of *fascism*, which is willing to achieve this through the use of force, to *populisms* in the other end, that wish to do this by way of legislation, distribution and a shrunken state power. The movements that have intended to destroy the middle classes have systematically failed and, since they eliminate the classes which are actually revolutionary, the middle classes, they are politically retrograde movements that stand in the opposite end of the material becoming of history.

It is unknown whether the dialectic process will ends with the destruction of capitalism. This is why the diagram poses questions about the future.

One thing to consider is, what does the order relation presented in the lattice mean when applied to social classes? It is clear that the notions of "true" or "false", in a logical sense, do not apply to social classes. However, the order relation does apply and there are at least two ways of interpreting it. One way is for $\geq$ to make reference to the *population quantity* in the class. In this situation, 0 would mean an empty set and 1, the total of human society. In this way, greater quantities correspond to the dominated classes–which are always the majority in terms

[182] This dialectic contradiction houses the major discussion of historic materialism and the varied "revisionisms' that took place in Marx's doctrine. These topics are out of the scope of this investigation.

[183] In rank-4 lattices, two middle classes can be distinguished that generate the new dominant and dominated classes, but the central class of free people does not exist. The central tier reappears in rank 5. These properties alternate between the even and odd ranks of the lattices considered.

[184] This statement contradicts the thesis of the *Communist Manifesto* [61] where it states that salaried workers will be the creators of a new society, contradicting the previous history. This topic is expanded in [32, 33, 34, 35, 36, 37].

of population–while the dominant classes would be the minority, Figure 27. But the opposite can also occur, as has been shown in Figure 34. In this case, $\geq$ measures *wealth* or *power* within a society, a parameter which is the inverse of the number of members. The middle classes, in both interpretations, lie somewhere in the middle.

These interpretations are quantitative and acceptable, but there is still a third way of identifying the order relation: by the value created by human work. In Marxist economy, the theory of value establishes that work creates a value that is equal for all the human beings that make up a society. This notion allows for a new definition of social classes.

The dominated classes are those who receive for their work less than the value created: they are exploited. Conversely, the dominant classes receive more than the value created–they are exploiters. The central class is the point where the sign changes and, for their work, its members receive the value that is effectively created. In summary, the order relation is given by the comparison of the retribution received for the work performed and the value created by this work. The point of balance–which approximately occurs in free workers–is the average social value created by work, or the global social average value per person. In essence, the three interpretations of the order relation are possible and ultimately equivalent.

These considerations and examples show that the main problem in applying dialectics to reality lies in determining the necessary lattice and the allocation of logical values to the actual statements considered.

## The formalism of historical materialism

A parallel between Figures 14 and 34 can be drawn immediately, allowing us to formalize the social classes present in the *Manifesto*:

| Figure 34 | Figure 14 |
|---|---|
| Roman owners | $E$ |
| Roman slaves | $a$ |
| Roman middle class | $p$ |
| Feudal noblemen | $A$ |
| Feudal serfs | $b$ |

| Feudal middle class | $q$ |
|---|---|
| Capitalist bourgeoisie | $B$ |
| Capitalist salaried workers | $c$ |
| Capitalist middle class | $r$ |

Class contradictions are expressed by means of the negation $N_{n-1}$: $E = N_{n-1}\, a$, $A = N_{n-1}\, b$, $B = N_{n-1}\, c$. The rotation of the central elements–for instance in **3D5**, see Figure 14–is expressed by means of the negation $N_0$: $p \to q$, $q \to r$ which allows us to interpret the succession of modes of reproduction when the equations of penetration of opposites are introduced, see Theorem 52.

Let us now consider the strict penetration of the pair $a \,\bar{*}\, E = p$, the proposition $p \to q$, which can be written as $a \,\bar{*}\, E \to q$, which establishes that these opposites "generate" or "produce" the central element $q$. Also, from $b \,\bar{*}\, A = q$ we have that $p \to q$, which can be expressed as $p \to b \,\bar{*}\, A$ establishing that the central element $p$ generates a new contradiction between classes.[185]

The major consequence that stems from these results is the existing difference between the notion of opposite classes–created by the negation $N_{n-1}$–and the becoming of the modes of production–created by the negation $N_0$–which show that these are *two different negations*. In other words, the contradiction between the dominant and the dominated classes is resolved because the middle classes create a new pair of opposing classes that are different from the original ones. This idea had already been introduced in [31]. This is one of the most important results of the formalization of dialectics.

Another result consists in explaining causality by becoming and accumulation in quantity. Let us consider a **3Dn** lattice with a large $n$. If

[185] In the interpretation of historical materialism, the first statement is translated as: *the contradiction between the dominated and the dominant class generates or creates an intermediate middle class.* The second statement establishes that: *the intermediate middle class generates or produces two new, opposing classes.* These statements contain two of the main theses of historical materialism–although they are not accepted by the Leninist interpretation–and are shown here as formal dialectic statements.

we use the mathematical notation, we would have:

$$d_i \bar{*} D_i = C_i \qquad N_0 \, C_i = C_{i+1}$$

From these results, we obtain a chain–as large as we want it to be–of processes of penetration of opposites which become new processes of the same type, a causal chain which permits, for instance, accumulation in quantity:

$$\cdots \to d_i \bar{*} D_i = C_i \to C_{i+1} = d_{i+1} \bar{*} D_{i+1} = C_{i+1} \to \cdots$$

This result applies to all circular accumulation evolutionary processes: the evolution of the species, the accumulation of money or the evolution of a human society.[186] The extended–and eventually infinite–causal cycle is an extension of the finite closed cycle which generates the negation $N_0$.

## Boundary cases

The various ramifications of the natural sciences lead to separate–and often contradictory–areas of knowledge, as has been shown in previous sections. This situation, which is unthinkable in scientific terms, intends to provide a unified vision of the universe. However, from a dialectic standpoint, nothing keeps "partial interpretations', contradictory among themselves, from existing. If we reject the notion of an absolute, true knowledge, and come to terms with the dialectic vision of reality, we will be faced with an entirely new set of problems.

The different "partial interpretations" imply that there may be a *boundary* between them. Many scientists have proposed this as an issue. It is possible that we may owe the first of these suggestions to Maxwell, who refers to the boundary between mechanics–back then, Newtonian mechanics, but this would also apply to quantum and relativistic mechanics–and everything that is encompassed by *statistical mechanics*.

Maxwell's idea consisted in imagining a "demon" that was capable of observing gas molecules, thus being able to distinguish between fast

[186] Put in humorous terms, this is the equation of the dilemma between the chicken and the egg.

and slow molecules in order to separate them. A floodgate joins two containers, **A** and **B** and allows the "demon" to permit a fast molecule coming from **A** towards the door, to enter container **B**. A similar thing would happen, but in reverse, with the slow molecules. This would cause the gas from container **B** to have a higher pressure and temperature than container **A**, thus violating the second principle of thermodynamics.

This case, as with all boundary cases, is an *intellectual experiment*: a being and a floodgate cannot coexist and interact with gas molecules. All the objects are formed by molecules of comparable size. But the actual possibility is not what is at stake here, but what happens in the frontier between classical, quantum or relativistic mechanics and statistical mechanics.

Fortunately, the problem of the boundary between the microscopic and the macroscopic description of a many-particle system can be analyzed in a quantitative manner. The number of particles establishes the properties in a known, precise relation:

> [ ... ] l*a fluctuation relative de toute grandeur additive $f$ décroît proportionellement à l'inverse de la racine carré du nombre de particules du corp macroscopique.*[187] [52, I, 2]

If we assume that a system formed by a number of particles $\sim 10^{20}$ has negligible fluctuation, a system with $\sim 10^{10}$ has an even greater fluctuation. The relation between both fluctuations is, according to the previous, $10^{10}/10^5 = 10^5$ that is, a hundred thousand times greater than known thermodynamics. Without a doubt, this "boundary case" does not fare well with either the microscopic or the macroscopic description. We are faced with a theory that has an intermediate behavior between the two, a theory which, for the time being, does not seem to be of any practical use. An answer to Maxwell's demon would only eventually occur within an intermediate theory between the macroscopic and the microscopic, one with an intermediate logical value, the penetration of values, for example, LH * MB from Figure 33.

[187] [ ... ] the relative fluctuation of every additive magnitude $f$ decreases proportionally to the inverse of the square root of the number of particles of the macroscopic body.

A second example of a boundary, similar to the previous, appears in Schrödinger's "cat". A container houses a cat, a flask of lethal gas and a radioactive source whose emission is capable of breaking the flask. This is a case of a boundary between microscopic and macroscopic mechanics, between quantum and classical physics. In the quantum vision, the flask is in an overlapping state between two others: the state of the intact flask and the state of the broken flask, given that it is possible that the emission of the radioactive particles may break it. The question, then, arises: is the cat dead or alive? Perhaps it is right at the overlapping state between dead and alive?

From a less spectacular point of view, we may ask this question: is there a specific size of a molecule or microscopic structure in which it begins to acquire macroscopic properties? We now know that there is such a thing as a giant molecule: carbon chains the likes of *nanotubes*, plastics or DNA. Nanotubes half a meter long can display amazing material properties. DNA molecules may be a few centimeters long. It seems that these giant molecules have at once quantum properties–such as the bonds between atoms–and classical properties–such as their dimensions or tensile strength. As in the previous case, the answer must be sought in an intermediate theory whose logical value is LH * MQ–see Figure 32.

A third classical example is found in the boundary between classical and relativistic mechanics. When approaching the speed of light, a "speed traveler" within a spaceship would experience phenomena related to the passage of time which are very different from what a normal traveler would experience. In this case, as the speed of light draws nearer, the spaceship would experience a change in its macroscopic physical properties because the crystalline mesh of the ship's metals, bound together by forces that propagate at the speed of light, would begin to change.[188] The same thing would occur to the traveler, who is nothing but a large biochemical machine with molecular bonds similar to those of metals. Possibly, a theory with logical value NM * RR could

[188] Let us assume that travel is taking place at *half the speed of light*. The relativistic correction factor is $1/\sqrt{1 - v^2/c^2} \approx 1.15$, then time, forces and other physical magnitudes experience a 15% modification, which is not negligible, for example, when dealing with material resistance or electronic systems.

provide an answer for the problem of the traveler; see Figure 32.

A fourth example–taken from real life, but which has never been formulated–is the case of a "giant land surveyor" taking it upon himself to convert the Amazon into a series of privately owned plots of land. This expert would come to discover that the measurements he takes soon become contradictory. For example, the sum of the angles of each triangle in his triangulation would add up to more than 180°. He would begin to experience the passage from Euclidean geometry to elliptical geometry on the surface of the Earth. This "land surveyor" would find use in a new legislation that would allow an elliptical description of geometry.

What do all of these examples teach us? We can draw some conclusions. The first is that the cases proposed *are not real*. They cannot exist since they hold within themselves an insurmountable contradiction. Maxwell's demon lives in two containers separated by a floodgate, but he is able to observe molecules. What material are the container, the door and the demon itself made of? If they were made up of molecules, the entire proposition would be absurd: the container as well as the door and even the demon would be meshes allowing molecules to pass through them, not being able to perform as expected. The same thing happens with Schrödinger's "cat": there is confusion between the scale of the cat and that of the particles that break the flask of poison. As she approaches the speed of light, the "fast traveler" would encounter many surprises. The material of the spaceship would cease to be functional–as would her own body–since all electric actions would undergo changes given that these actions also take place at the speed of light. To sum up, everything would shift until becoming impossible. The "giant land surveyor' would experience something similar. By attempting to triangulate the land, his instruments would prove to be useless due to the Earth's curvature. Prior to measuring an angle, he should create instruments belonging to the realm of elliptical geometry.

A second conclusion we can draw is that there are no imagined boundaries. As a phenomenon attempts to cross a boundary, it begins to change and the contradiction then fades. As the "demon" shrinks in size–to become for instance, about the size of the smallest insect–its neuron count shrinks, as well as its brain power and vision. Much prior to being able to handle gas molecules one by one, it would cease to exist as a living being. "Giant" molecules take no notice that they have abandoned quantum dimensions and those measurable by man. For the "speed traveler", the spaceship becomes a sort of cloud as it comes close to the speed of light; her bodily chemical reactions and navigation instruments will gradually stop working. She certainly will not need to worry about other relativistic effects. Something similar occurs with the "giant land surveyor" as he grows in size–not to mention how hard it will be for his bones to bear his own weight or his managing to breathe while in the upper layers of the atmosphere.

In summary, so-called boundaries are essentially inaccessible, as are the speed of light or an absolute zero. Regardless of this, the existence of the intermediate theories outlined in the various examples is still a possibility.

## Science and dialectics

The dialectic logic that has been formalized in this book, as we have shown, can be applied anywhere from mathematics to the social and natural sciences. It is an activity that we humans have spontaneously performed at least for as long as written records exist. It is a common element of daily life, the arts or humor.

The formalization of dialectics is an extension of Boole, Frege or Russell's binary logic. As such, it introduces new operations: the penetration of opposites, becoming or argumentation. It also extends the notions of negation and implication.

Dialectics display an essential difference between the formal and the experimental or social sciences. While the former are construed as universally valid, the latter are considered as having inferior logical value. What is more, they can never attain the quality of final truth that mathematics can.

In a natural manner, dialectic logic explains the existence of contradictory, yet valid and useful, scientific theories. It also accounts for the progression of science throughout history. Aristotle's mechanics established a certain value of truth. Galileo then improved upon the theory and Newton complemented it to a degree that seems to attain absolute truthfulness, an idea that upheld for a few centuries. $20^{th}$ century physicists–Einstein, Bohr, de Broglie, Heisenberg, Schrödinger and others–showed that there was another physics logically more valid than Newton's, although not without contradictions. The idea was born of a possible final unification of the entire physics–something hard to come to terms with, in dialectic terms.

A major conclusion of this study is that the construction of the homomorphism—the choice of lattice, negation and the association of statements on reality with the lattice elements—is not a systematic or mechanic process with precise rules. The application of dialectics to reality is a truly *creative process*–the rest is simply the application of formal rules. Just as in mathematics, the choice of universal truths–axioms–is the actual creative process. The demonstration of the resulting theorems, while a creative process in itself, is much less important from an epistemological standpoint.

# Bibliography


[1] Aitchison, Ian J. R.; MacManus, David A.; Snyder, Thomas M.*Understanding Heisenberg?s "magical" paper of July 1925: a new look at the calculational details*. 2008. Texto electrónico.

[2] Battro, Antonio M. *El pensamiento de Jean Piaget*. Emecé, Buenos Aires, 1969

[3] *King James Bible*.

[4] Birkhoff, Garrett. *Lattice Theory*. American Mathematical Society, New York, 1948.

[5] Birkhoff, Garrett. *Lattice Theory*. American Mathematical Society, Providence, 1967.

[6] Boole, George. *The Laws of Thought*. Dover Publications, Inc., New York, s/f.

[7] Bonvillain, Nancy. *The Zuni*. Texto electrónico en Gogle Books.

[8] Boyer, Carl B. Aristetle's Physics. Sc. Am. V. 182, N. 5, p. 48–51. May–1950.

[9] Catullus, Gaius. *Carmina*. Electronic text in *rudy.negenborn.net/catullus/*.

[10] Cavendish, Henry. *Experiments to Determine the Density of the Earth*. Phil. Trans. R. Soc. Lond. N. 88, p. 469–526, 1798. Electronic text .

[11] Carroll, Lewis *Symbolic Logic*. 1897. The Project Gutenberg EBook.

[12] Cervantes, Miguel de. *Don Quijote de la Mancha*. Real Academia Española, 2004.

[13] Chalmers, A. F. *What is this thing called Science?* Hackett Publishing Company, Inc. Indianapolis–Cambridge, 1999, 3th. edition.

[14] Comte, Auguste. *Cours de Philosophie Positive*. Bachelier, Libraire, Paris, 1830. 6 Vols.

[15] Darwin, Charles. *The origin of Species*. John Murray. London, 1897.

[16] Davey, B. A.; Priestley, H. A. *Introduction to Lattices and Order*. Cambridge University Press, 2002.

[17] Diogenes Laërtius. *Lives of the Eminent Philosophers*. Robert Drew Hicks trans. Electronic text.

[18] Dirac, Paul A. M. *The Principles of Quantum Mechanics*. Oxford University Press, 1958. Fourth Edition. Electronic text.

[19] Diccionario de la lengua española. Real Academia Española. Electronic text in *dle.rae.es*.

[20] *Empedokles of Akragas*, John Burnet. Electronic text in *www.classicpersuasion.org/pw/burnet/*.

[21] Engels, Friedrich. *Dialektik der Natur*. 1873–1883. Electronic text in *www.marxists.org*.

[22] Engels, Friedrich. *Anti–Dühring*. Leipzig, 1878. Electronic text in *www.marxists.org*.

[23] Eudlid. *The Thirteen Books of Euclid's Elements*. Trad. and coment. Sir Thomas Heath. $2^{ed.}$. Dover Publications. Inc. New York, s/f. 3 Vols.

[24] Fedak, William A.; Prentis, Jeffrey J. *The 1925 Born and Jordan paper "On quantum mechanics"*. Am. J. Phys. N. 77, V. 2, February 2009. Electronic text.

[25] Fitch, F. B. *Symbolic Logic. An introduction*. The Ronald Press Company, New York, 1952.

[26] Freud, Sigmund. *Three Essays on the Theory of Sexuality*. Electronic text in *www.sigmundfreud.net*.

[27] Freud, Sigmund. *Jokes and Their Relation to the Unconscious*. James Strachey, Trans. The Hogarth Press, London, 1960. Complete Works, Vol. VIII. Electronic text.

[28] García Bacca, J. D. *Los Presocráticos*. Traductor y comentarista. 2 vols. El Colegio de México, México, 1943.

[29] Gödel, Kurt. *On Formally Undecidable Proposition of Principia Mathematica and related Systems*. 1931. In *From Frege to Gödel*, Van Herjensort Editor. p. 596–616.

[30] Granet, Marcel. *La Religion des Chinois*. Paris, 1922.

[31] Grompone, Juan. *Sur la dynamique des classes dans le "Manifeste Communiste"*. Le manifeste communiste 150 ans après. Rencontre Internationale. Paris, 1998.

[32] Grompone, Juan. *El nacimiento de las sociedades de clase*. "La Danza de Shiva", Libro II. Fin de Siglo, Montevideo, 2013.

[33] Grompone, Juan. *Las sociedades feudales*. "La Danza de Shiva", Libro II. La flor del Itapebí, Montevideo, 2009.

[34] Grompone, Juan. *Las sociedades esclavistas*. "La Danza de Shiva", Libro III. (En preparación.)

[35] Grompone, Juan. *La sociedad capitalista*. "La Danza de Shiva", Libro IV. (En preparación.)

[36] Grompone, Juan. *La construcción del futuro*. "La Danza de Shiva", Libro V. Fin de Siglo, Montevideo, 2014.

[37] Grompone, Juan. *Materialismo Histórico*. La danza de Shiva VI. (En preparación.)

[38] Grompone, Juan. *La Lógica de la Poesía en la Obra de Heine*. Homenaje al poeta Enrique Heine. Montevideo, 1973.

[39] Guzmán de Rojas, Iván. *Problemática Lógico–Lingüística de la Comunicación Social con el pueblo Aymara*. Otawa, 1982.

[40] Harrington, John Peabody. *The Ethnogeography of the Tewa Indians*. Washington, Government Printing Office, 1916. Electronic text in *archive.org*.

[41] Hayes, J. P. *A Unified Switching Theory with Applications to VLSI Design*. Proc. IEEE, V. 70, N. 10, p.1140–1151, Oct 1982.

[42] Hegel, G. W. F. *Science de la Logique*. 3 Vols, Paris, 1981.

[43] Heine, Heinrich. *Buch der Lieder – Livre des Chants*. Aubier, Paris, 1947, 2 Vols.

[44] Heisenberg, Werner. *Physics and Philosophy: the revolution in modern science*. Harper & Brothers, New York, 1962.

[45] Heraclitus. *The Complete Fragments*. Translation, Commentary and the Greek text by William Harris, Middlebury College. Electronic text.

[46] Hilbert, David. *Fundamentos de la geometría*. Publicaciones del Instituto "Jorge Juan" de matemáticas. Madrid, 1953.

[47] Heisenberg, W. *Quantum–theoretical Re–interpretation of Kinematic and Mechanical Relations*. Zeitschrift für Physik, 1925. Electronic text.

[48] Hooker, C. A. (editor) *The Logico–Algebraic Approach to Quantum Mechanics*. 2 vols., Ontario, 1975, 1979.

[49] Hughes, R. I. G. *Quantum Logic*. Sc. Amer., V. 243, N. 4, p. 146–157, Oct 1981.

[50] Kepler, Johannes. *Armonice Mundi*. En *Ptolemy, Copernicus, Kepler*, Great Books of the Western World, Robert Maynard Hutchins, Editor In Chief. Encyclopedia Britannica, Inc. 1952. Electronic text.

[51] Landau, Lev; Lifchitz, Evgeny. *Mécanique*. Edition en Langues Etrangères, Moscou, 1960.

[52] Landau, Lev; Lifchitz, Evgeny. *Physique Statistique*. Edition Mir, Moscou, 1967.

[53] Lao Tse. *Tao–Te–King*. (Versión de Paolo Siao Sci–Yi) Bari, 1947.

[54] Lavoisier, Antoine–Laurent de. *Mémoire sur la nature du principe qui se combine avec les métaux pendant leur calcination et qui en augmente le poids*. Mémoires de l'Académie des sciences, 1775, p. 520. Electronic text.

[55] Lenin, V. I. *Materialism and Empirio-criticism*. Trans. Abraham Fineberg, Moscow, 1972. Electronic text in *www.marxists.org/archive/lenin/*.

[56] Levendel, Y. H.; Menon, P. R. *Test Generation Algorithms for Computer Hardware Description Languages*. IEEE Trans. Comp., C–31, p. 577–588, Jul 1982.

[57] Lukasiewicz, J.; Tarski, A. *Untersuchungen über den Aussangenkalkul*. Comp. Rend. Acad. Vars., p. 30–50, 1930.

[58] Lukasiewicz, J. *Philosophische Bemerkungen zu Mehrwertigen Systemen des Aussagenkalkuls*. Comp. Rend. Acad. Vars., p. 51–77, 1930.

[59] Mao Tse–tung. *Sobre la contradicción*. Ediciones en lenguas extranjeras, Pekín, 1965.

[60] Marx, Karl. *Das Kapital*. Europäische Verlagsanstalt. Leipzig, 1968. 6 Vols. Electronic text in *www.marxists.org*.

[61] Marx, Karl; Engels, Friedrich. *Manifest der Kommunistischen Partei*. Mursia, Milano, 1984. Electronic text in *www.marxists.org*.

[62] Marx, Karl. *Miseria de la Filosofía*. Buenos Aires, 1971. Electronic text in *www.marxists.org*.

[63] Marx, Karl. *Capital. A Critique of Political Economy.* Trans. Samuel Moore and Edward Aveling, First English edition, 1887. Electronic text in *www.marxists.org*.

[64] Massera, José Luis. *Dialéctica y matemática*. Facultad de Humanidades y Ciencias, Montevideo, 1985. Electronic text.

[65] Massera, José Luis. *Reflexiones de un matemático sobre la dialéctica*. Universidad de la República, Montevideo, 1998. Electronic text.

[66] Morgan, Lewis Henry. *Ancient Society*. Charles H. Kerr & Company. Chicago, 1877.

[67] Newton, Ys. *Philosophiæ Naturalis Principia Mathematica*. Londini, MDCLXXXVIII (1686). Ed. facsimilar, Dawson & Sons Ltd., London, s.f.

[68] Newtono, Isaaco. *Philosophiæ Naturalis Principia Mathematica. Editio tertia aucta & emendata.* Londini, MDCCXXVI (1726). Electronic text.

[69] Newton, Isaac. *Mathematical Principles of Natural Philosophy and his System of the World*. Andrew Motte translator. Florian Cajori revised. University of California Press, 1962, 2 Vols.

[70] Newton, Isaac. *The Principia: mathematical principles of natural philosophy*, trans. I. B. Cohen and A. Whitman, Berkeley: University of California Press, 1999.

[71] Newton, Isaac. *Opticks*. Fourth edition, London, 1730. Dover Publications Inc. New York, 1952.

[72] *New Oxford American Dictionary*. Versión 2.2.1, Electronic text.

[73] Paracelsus the Great. *The Hermetic and Alchemical Writings*. Arthur Edward Waite, editor. Electronic text in Google Books.

[74] Petrarca, Francesco. *Rime – Trionfe e Poesie Latine*. Ricciadrdi Editore, Verona,1951,

[75] Piaget, Jean. *Essai de Logique Opératoire*. Paris, 1972.

[76] Piaget, Jean. *Las formas elementales de la dialéctica*. Gedisa, Barcelona, 1982.

[77] Poincaré, Henri. *La Science et l'Hypothèse*. Flammarion, Paris, 1902. Electronic text.

[78] Poincaré, Henri. *La Valeur de la Science*. Flammarion, Paris, 1905. Electronic text.

[79] Poincaré, Henri. *Science et Méthode*. Flammarion, Paris, 1908. Electronic text.

[80] Poincaré, Henri. *Dernières Pensées*, Flammarion, Paris, 1913, 1926. Electronic text.

[81] Poincaré, Henri. *Les Méthodes Nouvelles de la Mécanique Céleste.* Gauthier–Villars, Paris, 1892.

[82] Post, E. L. *Introduction to a General Theory of Elementary Propositions*. Am. Jour. Math., T. 43, p. 163–185, 1921.

[83] Quine, W. V. *Paradox*. Sc. Amer., V 206, N 4, p. 84–96, Apr 1962.

[84] Lorentz, H. A., Einstein, A., Minkowski, H., Weil, H. *The Principle of Relativity*. Dover Publications Inc. s/f

[85] Russell, Bertrand. *History of Western Philosophy*. London, 1961.

[86] Russell, Bertrand, Whitehead, Alfred North. *Principia Mathematica*. Second Edition, Reprinted, Cambridge, 1935, 3 Vols.

[87] Russell, Bertrand. *The Problems of Philosophy*. Prometheus Books, Buffalo, New York, 1988.

[88] Saussure, Ferdinand de. *Cours de Linguistique Générale*. Payot, Paris. 1940.

[89] Schrödinger, E. *An Undulatory Theory of the Mechanics of Atoms and Molecules*. The Physical Review, S. 2, V. 28, N. 6, Dec–1926. Electronic text.

[90] Smith, K. C. *The Prospects for Multivalued Logic*. IEEE Trans. Comp., C–30, p. 619–634, Sep 1981,

[91] Thomason, Richmond H. *Symbolic Logic. An Introduction*. Macmillan Company, London, 1970.

[92] Toynbee, Arnold J. *A Study of History*. Oxford University Press, London, 1948. 6 Vols.

[93] Toynbee, Arnold J. *A Study of History*. (Abridgement of Volumes I–VI by D.C. Somervell). Oxford University Press, New York, 1947.

[94] Vico, Giambattista. *La scienza nuova*. Letteratura italiana Einaudi, Milano, 1959. Electronic text.

[95] Wagner, Richard. *Tristan und Isolde*. 1857.

[96] Wilde, Oscar, *Intentions. The Truth of Masks.* 1891. Electronic text in Google Books.

[97] Zuangzi. *The Sacred Books of China: The Texts of Taoism, Part I*. Oxford: Oxford University Press, Legge, James translator, 1891. Electronic text.

[98] Zuangzi. *Chuang Tzu: Basic Writings*. Watson, Burton translator. Columbia University Press, New York, 1964. Electronic text.

[99] Zhuangzi. Translated by Nina Correa. Electronic text.

[100] Zhuangzi. *The Writings of Chuang Tzu*, James Legge, 1891. Electronic text in Chinese Text Project.

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The purpose of the present analytical index is to assist reading. While it does not include all the proper names, concepts or geographical names mentioned in the book, it is useful in helping to locate the main ideas.

This index has a logical structure, which makes it somewhat difficult to search for an entry. However, it will certainly be helpful to readers who choose to go through the text without following the predefined order. It is possible for seemingly unrelated concepts to be associated here. The most important references appear in bold.

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